4.17 Use the first derivative test to find all local extrema for $f(x) = \sqrt[3]{x-1}$.

Concavity and Points of Inflection

We now know how to determine where a function is increasing or decreasing. However, there is another issue to consider regarding the shape of the graph of a function. If the graph curves, does it curve upward or curve downward? This notion is called the **concavity** of the function.

Figure 4.34(a) shows a function f with a graph that curves upward. As x increases, the slope of the tangent line increases. Thus, since the derivative increases as x increases, f' is an increasing function. We say this function f is concave up. Figure 4.34(b) shows a function f that curves downward. As x increases, the slope of the tangent line decreases. Since the derivative decreases as x increases, f' is a decreasing function. We say this function f is concave down.

Definition

Let f be a function that is differentiable over an open interval I. If f' is increasing over I, we say f is **concave up** over I. If f' is decreasing over I, we say f is **concave down** over I.

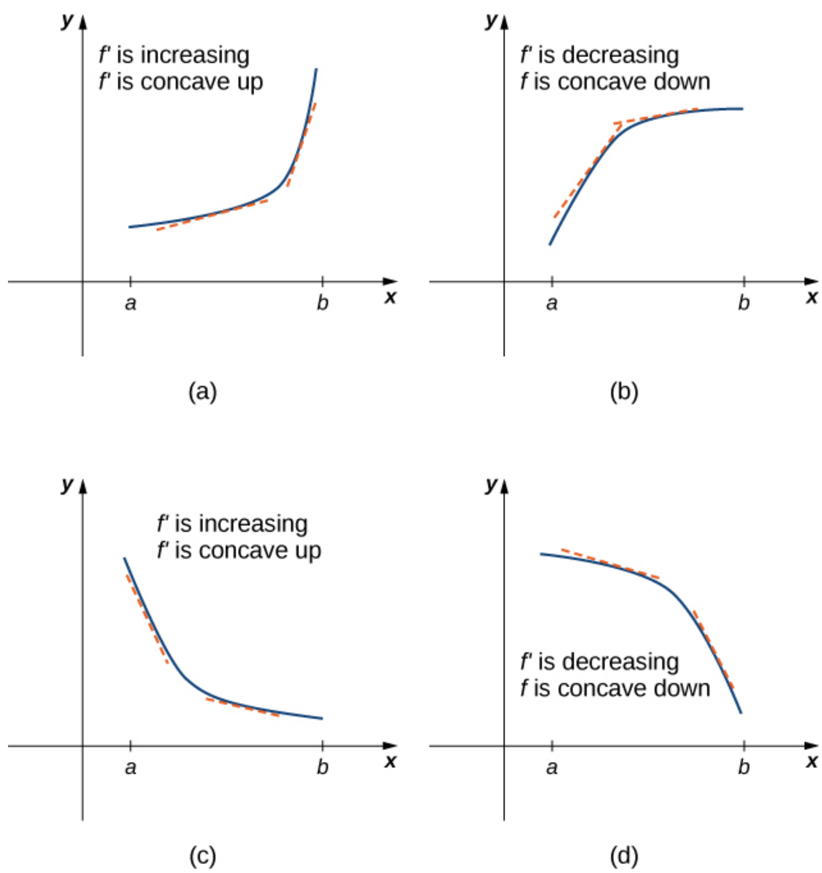

Figure 4.34 (a), (c) Since f' is increasing over the interval (a, b), we say f is concave up over (a, b). (b), (d) Since f' is decreasing over the interval (a, b), we say f is concave down over (a, b).

In general, without having the graph of a function f, how can we determine its concavity? By definition, a function f is concave up if f' is increasing. From Corollary 3, we know that if f' is a differentiable function, then f' is increasing if its derivative $f''(x) > 0$. Therefore, a function f that is twice differentiable is concave up when $f''(x) > 0$. Similarly, a function f is concave down if f' is decreasing. We know that a differentiable function f' is decreasing if its derivative $f''(x) < 0$. Therefore, a twice-differentiable function f is concave down when $f''(x) < 0$. Applying this logic is known as the **concavity test**.

Theorem 4.10: Test for Concavity

Let f be a function that is twice differentiable over an interval I.

i. If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.

ii. If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

We conclude that we can determine the concavity of a function f by looking at the second derivative of f. In addition, we observe that a function f can switch concavity (**Figure 4.35**). However, a continuous function can switch concavity only at a point x if $f''(x) = 0$ or $f''(x)$ is undefined. Consequently, to determine the intervals where a function f is concave up and concave down, we look for those values of x where $f''(x) = 0$ or $f''(x)$ is undefined. When we have determined these points, we divide the domain of f into smaller intervals and determine the sign of f'' over each of these smaller intervals. If f'' changes sign as we pass through a point x, then f changes concavity. It is important to remember that a function f may not change concavity at a point x even if $f''(x) = 0$ or $f''(x)$ is undefined. If, however, f does change concavity at a point a and f is continuous at a, we say the point $(a, f(a))$ is an inflection point of f.

Definition

If f is continuous at a and f changes concavity at a, the point $(a, f(a))$ is an **inflection point** of f.

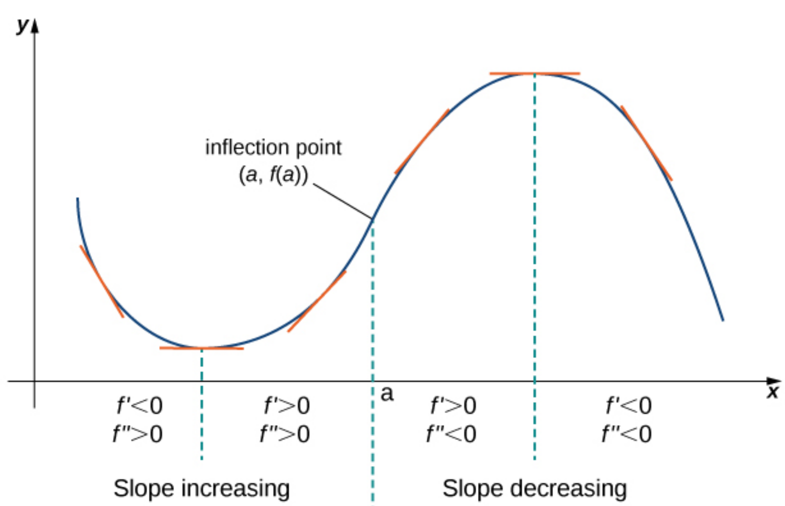

Figure 4.35 Since $f''(x) > 0$ for $x < a$, the function f is concave up over the interval $(-\infty, a)$. Since $f''(x) < 0$ for $x > a$, the function f is concave down over the interval (a, ∞). The point $(a, f(a))$ is an inflection point of f.

Example 4.19

Testing for Concavity

For the function $f(x) = x^3 - 6x^2 + 9x + 30,$ determine all intervals where f is concave up and all intervals where f is concave down. List all inflection points for f. Use a graphing utility to confirm your results.

Solution

To determine concavity, we need to find the second derivative $f''(x).$ The first derivative is $f'(x) = 3x^2 - 12x + 9,$ so the second derivative is $f''(x) = 6x - 12.$ If the function changes concavity, it occurs either when $f''(x) = 0$ or $f''(x)$ is undefined. Since f'' is defined for all real numbers $x,$ we need only find where $f''(x) = 0.$ Solving the equation $6x - 12 = 0,$ we see that $x = 2$ is the only place where f could change concavity. We now test points over the intervals $(-\infty, 2)$ and $(2, \infty)$ to determine the concavity of f. The points $x = 0$ and $x = 3$ are test points for these intervals.

Interval	**Test Point**	**Sign of $f''(x) = 6x - 12$ at Test Point**	**Conclusion**
$(-\infty, 2)$	$x = 0$	$-$	f is concave down
$(2, \infty)$	$x = 3$	$+$	f is concave up.

We conclude that f is concave down over the interval $(-\infty, 2)$ and concave up over the interval $(2, \infty).$ Since f changes concavity at $x = 2,$ the point $(2, f(2)) = (2, 32)$ is an inflection point. Figure 4.36 confirms the analytical results.

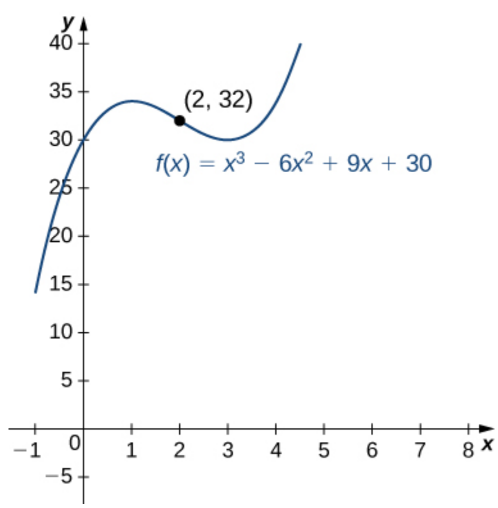

Figure 4.36 The given function has a point of inflection at $(2, 32)$ where the graph changes concavity.

4.18 For $f(x) = -x^3 + \frac{3}{2}x^2 + 18x,$ find all intervals where f is concave up and all intervals where f is concave down.

We now summarize, in **Table 4.6**, the information that the first and second derivatives of a function f provide about the graph of $f,$ and illustrate this information in **Figure 4.37**.

Sign of f'	Sign of f''	Is f increasing or decreasing?	Concavity
Positive	Positive	Increasing	Concave up
Positive	Negative	Increasing	Concave down
Negative	Positive	Decreasing	Concave up
Negative	Negative	Decreasing	Concave down

Table 4.6 What Derivatives Tell Us about Graphs

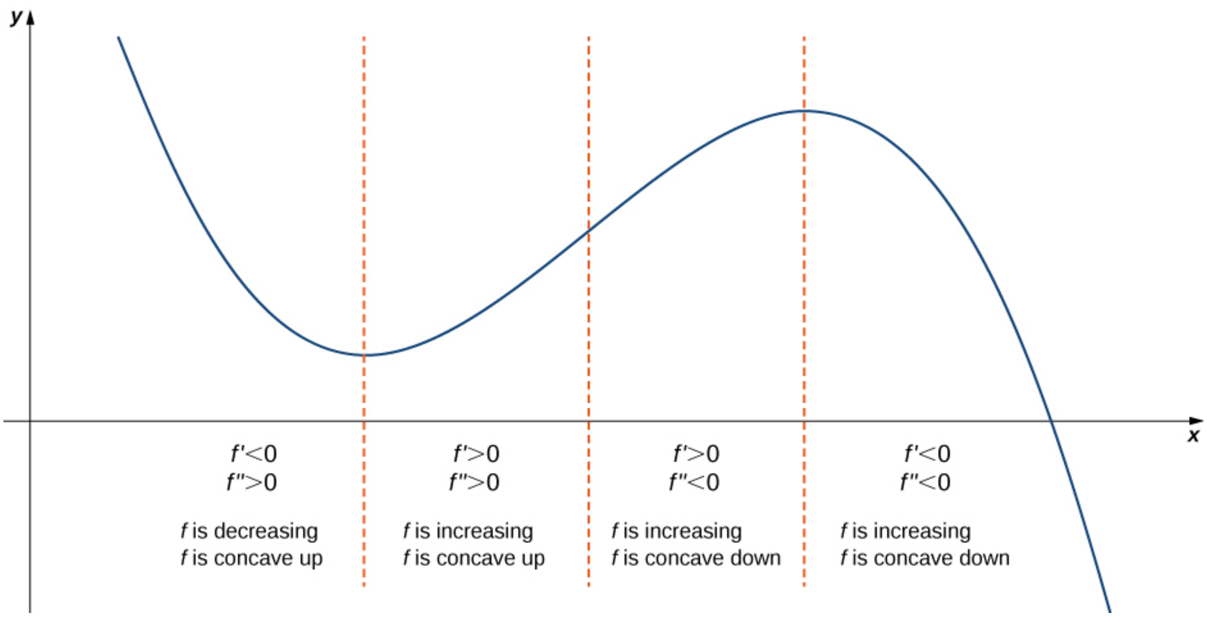

Figure 4.37 Consider a twice-differentiable function f over an open interval $I.$ If $f'(x) > 0$ for all $x \in I,$ the function is increasing over $I.$ If $f'(x) < 0$ for all $x \in I,$ the function is decreasing over $I.$ If $f''(x) > 0$ for all $x \in I,$ the function is concave up. If $f''(x) < 0$ for all $x \in I,$ the function is concave down on $I.$

The Second Derivative Test

The first derivative test provides an analytical tool for finding local extrema, but the second derivative can also be used to locate extreme values. Using the second derivative can sometimes be a simpler method than using the first derivative.

We know that if a continuous function has a local extrema, it must occur at a critical point. However, a function need not have a local extrema at a critical point. Here we examine how the **second derivative test** can be used to determine whether a function has a local extremum at a critical point. Let f be a twice-differentiable function such that $f'(a) = 0$ and f'' is continuous over an open interval I containing $a.$ Suppose $f''(a) < 0.$ Since f'' is continuous over $I,$ $f''(x) < 0$ for all $x \in I$ (**Figure 4.38**). Then, by Corollary 3, f' is a decreasing function over $I.$ Since $f'(a) = 0,$ we conclude that for all $x \in I$, $f'(x) > 0$ if $x < a$ and $f'(x) < 0$ if $x > a$. Therefore, by the first derivative test, f has a local maximum at $x = a$. On the other hand, suppose there exists a point b such that $f'(b) = 0$ but $f''(b) > 0$. Since f'' is continuous over an open interval I containing b, then $f''(x) > 0$ for all $x \in I$ (**Figure 4.38**). Then, by Corollary 3, f' is an increasing function over I. Since $f'(b) = 0$, we conclude that for all $x \in I$, $f'(x) < 0$ if $x < b$ and $f'(x) > 0$ if $x > b$. Therefore, by the first derivative test, f has a local minimum at $x = b$.

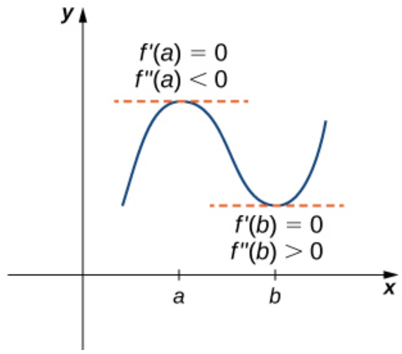

Figure 4.38 Consider a twice-differentiable function f such that f'' is continuous. Since $f'(a) = 0$ and $f''(a) < 0$, there is an interval I containing a such that for all x in I, f is increasing if $x < a$ and f is decreasing if $x > a$. As a result, f has a local maximum at $x = a$. Since $f'(b) = 0$ and $f''(b) > 0$, there is an interval I containing b such that for all x in I, f is decreasing if $x < b$ and f is increasing if $x > b$. As a result, f has a local minimum at $x = b$.

Theorem 4.11: Second Derivative Test

Suppose $f'(c) = 0$, f'' is continuous over an interval containing c.

i. If $f''(c) > 0$, then f has a local minimum at c.

ii. If $f''(c) < 0$, then f has a local maximum at c.

iii. If $f''(c) = 0$, then the test is inconclusive.

Note that for case iii. when $f''(c) = 0$, then f may have a local maximum, local minimum, or neither at c. For example, the functions $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all have critical points at $x = 0$. In each case, the second derivative is zero at $x = 0$. However, the function $f(x) = x^4$ has a local minimum at $x = 0$ whereas the function $f(x) = -x^4$ has a local maximum at x, and the function $f(x) = x^3$ does not have a local extremum at $x = 0$.

Let’s now look at how to use the second derivative test to determine whether f has a local maximum or local minimum at a critical point c where $f'(c) = 0$.

Example 4.20

Using the Second Derivative Test

Use the second derivative to find the location of all local extrema for $f(x) = x^5 - 5x^3.$

Solution

To apply the second derivative test, we first need to find critical points c where $f'(c) = 0.$ The derivative is $f'(x) = 5x^4 - 15x^2.$ Therefore, $f'(x) = 5x^4 - 15x^2 = 5x^2\left(x^2 - 3\right) = 0$ when $x = 0, \pm\sqrt{3}.$

To determine whether f has a local extrema at any of these points, we need to evaluate the sign of f'' at these points. The second derivative is

$$f''(x) = 20x^3 - 30x = 10x\left(2x^2 - 3\right).$$

In the following table, we evaluate the second derivative at each of the critical points and use the second derivative test to determine whether f has a local maximum or local minimum at any of these points.

x	$f''(x)$	Conclusion
$-\sqrt{3}$	$-30\sqrt{3}$	Local maximum
0	0	Second derivative test is inconclusive
$\sqrt{3}$	$30\sqrt{3}$	Local minimum

By the second derivative test, we conclude that f has a local maximum at $x = -\sqrt{3}$ and f has a local minimum at $x = \sqrt{3}.$ The second derivative test is inconclusive at $x = 0.$ To determine whether f has a local extrema at $x = 0,$ we apply the first derivative test. To evaluate the sign of $f'(x) = 5x^2\left(x^2 - 3\right)$ for $x \in \left(-\sqrt{3}, 0\right)$ and $x \in \left(0, \sqrt{3}\right),$ let $x = -1$ and $x = 1$ be the two test points. Since $f'(-1) < 0$ and $f'(1) < 0,$ we conclude that f is decreasing on both intervals and, therefore, f does not have a local extrema at $x = 0$ as shown in the following graph.

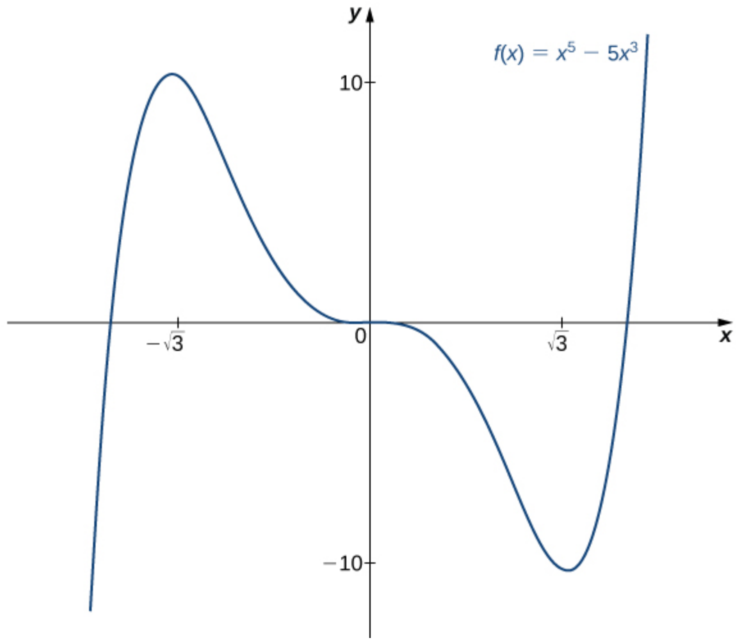

Figure 4.39 The function f has a local maximum at $x = -\sqrt{3}$ and a local minimum at $x = \sqrt{3}$

4.19 Consider the function $f(x) = x^3 - \left(\frac{3}{2}\right)x^2 - 18x$. The points $c = 3, -2$ satisfy $f'(c) = 0$. Use the second derivative test to determine whether f has a local maximum or local minimum at those points.

We have now developed the tools we need to determine where a function is increasing and decreasing, as well as acquired an understanding of the basic shape of the graph. In the next section we discuss what happens to a function as $x \to \pm\infty$. At that point, we have enough tools to provide accurate graphs of a large variety of functions.