The Virial Theorem

The Virial Theorem is a very important theorem in mechanics and can be applied to many areas of Astrophysics, including stellar evolution (gravitational contraction energy source) and mass determination for gravitationally bound systems such as galaxy clusters.

Consider a system of n point particles indexed by i. Let $\vec{r}_i$, $\vec{v}_i$ and $\vec{p}_i$ be the position, velocity and momentum vectors, respectively, for the i-th particle. Its mass is denoted as m_i . Let $\vec{F}_i$ be the net force, internal and external, on the i-th particle.

Statement of the Virial Theorem:

For the n point particles bound together into a system the time average of the kinetic energy of the particles, $\sum \frac{1}{2} m v_i^2$, plus one half of the time average of $\sum \vec{F}_i \cdot \vec{r}_i$ is equal to zero.

Proof:

$$\vec{v}_i = \frac{d\vec{r}_i}{dt} \qquad \vec{p}_i = m_i \vec{v}_i \qquad \frac{d\vec{p}_i}{dt} = \vec{F}_i$$

$$H = \sum \vec{p}_i \cdot \vec{r}_i$$

$$\frac{dH}{dt} = \sum \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i + \sum \vec{p}_i \cdot \left(\frac{d\vec{r}_i}{dt}\right)$$

$$\sum \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i = \sum \vec{F}_i \cdot \vec{r}_i$$

$$\sum \vec{p}_i \cdot \left(\frac{d\vec{r}_i}{dt}\right) = \sum m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 = \sum m_i v_i^2$$

$$\sum m_i v_i^2 = 2K \qquad (K = KE \text{ of the system})$$

$$\frac{dH}{dt} = \sum \vec{F}_i \cdot \vec{r}_i + 2K$$

Time average of the variables: $$\bar{f} = \frac{1}{\tau}\int_0^\tau f(t)\,dt$$

$$\overline{\frac{dH}{dt}} = \overline{\sum \vec{F}_i \cdot \vec{r}_i} + 2\overline{K}$$

$$\overline{\frac{dH}{dt}} = \frac{1}{\tau}\int_0^\tau \frac{dH}{dt}\,dt = \frac{1}{\tau}\left[H(\tau) - H(0)\right]$$

If the system is cyclical such that it returns to its initial state after an interval then can be chosen equal to the cycle period and $\overline{\frac{dH}{dt}}$ reduces to zero. If the system is not cyclical, but is bounded such that $\left[H(\tau) - H(0)\right]$ is finite for any τ the limit of $\overline{\frac{dH}{dt}}$ as τ increases without bound is zero.

$$\sum \overline{\vec{F}_i \cdot \vec{r}_i} + 2\overline{K} = 0 \qquad \overline{K} + \frac{1}{2}\sum \overline{\vec{F}_i \cdot \vec{r}_i} = 0$$

if the forces are conservative, then: $\vec{F}_i = -\frac{\vec{\partial}}{\partial r_i} V$

and: $\overline{K} - \frac{1}{2}\overline{\sum \frac{\vec{\partial}}{\partial r_i} V \cdot \vec{r}_i} = 0$

For the inverse square law force, using Euler's Theorem for homogeneous functions of degree -1 (potential functions of gravity or electrostatic forces):

$$\overline{\sum \frac{\vec{\partial}}{\partial r_i} V \cdot \vec{r}_i} = -\overline{V}$$

$$\overline{K} + \frac{1}{2}\overline{V} = 0 \qquad \boxed{\overline{K} = -\frac{1}{2}\overline{V}}$$

For bound gravitational systems the potential energy is negative so the KE is positive.

The total average energy of the system $\overline{T} = \overline{K} + \overline{V}$ is:

$$\boxed{\overline{T} = \frac{1}{2}\overline{V}}$$