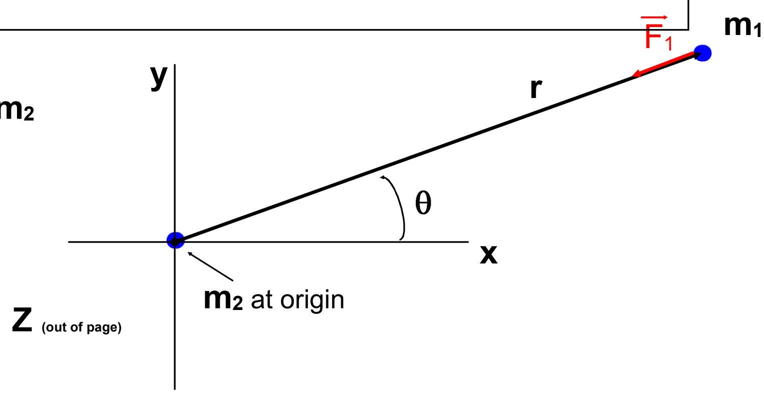
Solution to motion under a central force (inverse square law):

$$\boxed{m_1\ddot{\vec{r}} = -\frac{k}{r^2}\hat{e}_r \quad \text{where } k = G(m_1+m_2)m_1}$$

motion of $\mathbf{m_1}$ relative to $\mathbf{m_2}$



Since $\vec{L}$ conserved ⟶ motion in a plane ⟶ use cylindrical/polar coordinates

unit vectors: $\hat{e}_r \quad \hat{e}_\theta \quad \hat{e}_z$

Write the D.E. (top of page) in cyl/polar coordinates:

Note, from vector calculus:

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{e}_\theta$$

The vector equation at the top of the page can now be written as 2 scalar DE's, one for each polar component:

$$\hat{e}_r: \quad m_1(\ddot{r} - r\dot{\theta}^2) = -\frac{k}{r^2}$$

$$\hat{e}_\theta: \quad m_1(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

Solve to get **r = r(t)** **θ = θ(t)**
then eliminate **t** to get **r = r(θ)** for shape of orbit:

One possible solution is an elliptical orbit with one focus at the origin (Law of Ellipses, Kepler's 1st law)

Solve the "θ" equation for Kepler's 2nd law (Law of Areas):

$$m_1(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

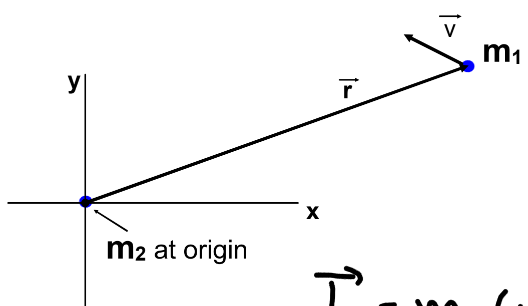
Note: $\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{\theta}\dot{r} + r^2\ddot{\theta}$

$= r(2\dot{\theta}\dot{r} + r\ddot{\theta})$

Then: $m_1\frac{d}{dt}(r^2\dot{\theta}) = 0 \qquad \frac{d}{dt}(r^2\dot{\theta}) = 0$

$$r^2\dot{\theta} = l \qquad l = \text{const}$$

What is l ? Use definition of angular momentum ($\vec{L}$) to investigate



$$\vec{L} = m_1(\vec{r} \times \vec{V})$$

Written in cylindrical coordinates:

$$\vec{V} = V_r\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

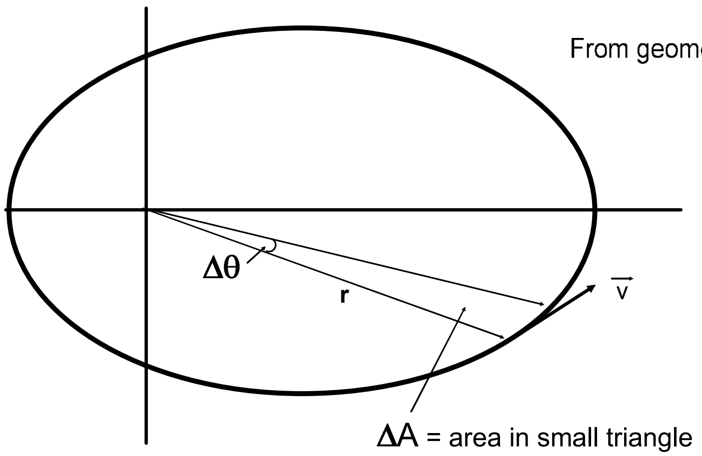
$$\vec{L} = m_1(r\hat{e}_r \times (V_r\hat{e}_r + r\dot{\theta}\hat{e}))$$

$$\vec{L} = m_1(rV_r\underbrace{(\hat{e}_r \times \hat{e}_r)}_{0} + r^2\dot{\theta}\underbrace{(\hat{e}_r \times \hat{e}_\theta)}_{e_z})$$

$$\vec{L} = m_1 r^2\dot{\theta}\,\hat{e}_z$$

So: $l = r^2\dot{\theta} = \frac{L}{m_1}$

$l \equiv$ angular momentum per unit mass



From geometry of ellipse:

$$\Delta A = \frac{1}{2} r (r \Delta\theta)$$

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t} \longrightarrow \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} \equiv \text{"areal velocity"}$$

$$r^2 \dot{\theta} = l \quad \text{and} \quad l = \text{const} \longrightarrow \frac{dA}{dt} = \frac{1}{2} l = \text{const}$$

Equal areas in equal time intervals: Law of Areas

General solution (after eliminating **t**):

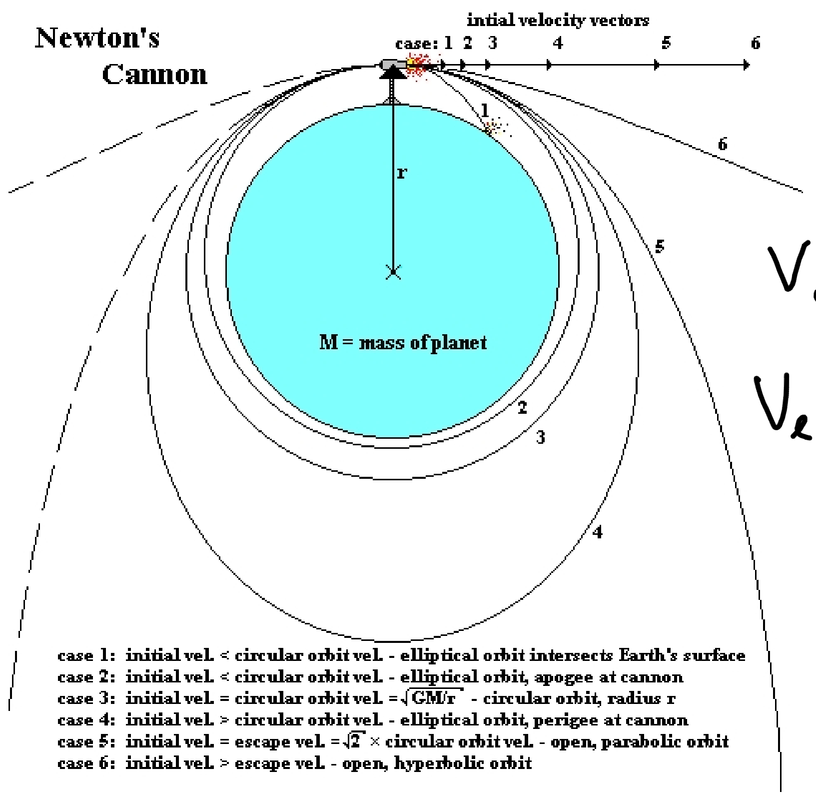
$$r = \frac{1}{A\cos\theta + \frac{k}{m_1 l^2}} \qquad \left(l = \frac{L}{m_1} \equiv \text{orbital } L \text{ per unit mass} \right)$$

Compare to the polar equation for conic sections:

$$r = \frac{1}{\frac{e}{a(1-e^2)}\cos\theta + \frac{1}{a(1-e^2)}}$$

Or:

$$\left(r = \frac{a(1-e^2)}{1 + e\cos\theta} \right)$$



$$V_{cir} = \sqrt{\frac{GM}{r}}$$

$$V_{esc} = \sqrt{2}\, V_{cir}$$