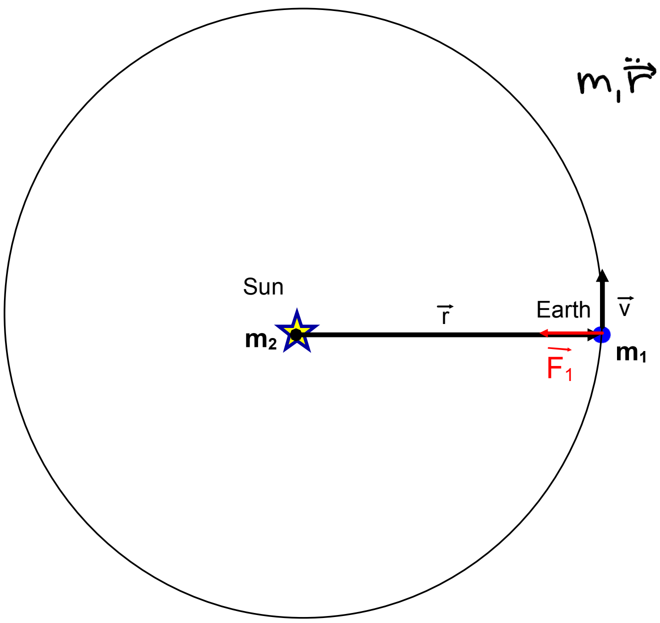
Newton's version of Kepler's 3rd law from the special case of motion of a planet in a circular orbit around the Sun.



$$m_1\ddot{\vec{r}} = -\frac{k}{r^2}\hat{e}_r \quad \text{where } k = G(m_1+m_2)m_1$$

$$\ddot{\vec{r}} = \vec{a}_{centripetal}$$

$$a_{centripetal} = \frac{v_1^2}{r}$$

$$-\frac{m_1 v_1^2}{r} = -\frac{k}{r^2} \qquad v_1 = \frac{2\pi r}{\tau} \left(\tau = \text{period of orbit}\right)$$

$$\frac{m_1\left(\frac{2\pi r}{\tau}\right)^2}{r} = \frac{k}{r^2} \qquad \frac{\cancel{m_1}\,4\pi^2 r^2}{\cancel{r}\,\tau^2} = \frac{G(m_1+m_2)\cancel{m_1}}{r^{\cancel{2}}}$$

$$\boxed{\frac{\tau^2}{r^3} = \frac{4\pi^2}{G(m_1+m_2)}}$$

Newton's version of Kepler's 3rd law from the more general case of motion of a planet in an elliptical orbit around the Sun.

From conservation of angular momentum: $\frac{dA}{dt} = \frac{1}{2}\ell \qquad \ell = \frac{L}{m_1}$

Note: $A \equiv$ area

$$A = \int_0^\tau \frac{dA}{dt}\,dt = \int_0^\tau \frac{1}{2}\ell\,dt = \frac{1}{2}\ell\tau$$

From geometry:

Area for an ellipse

$$A = \pi ab \qquad b = a\sqrt{1-e^2} \longrightarrow A = \pi a^2\sqrt{1-e^2}$$

From the solution for an orbit in polar coordinates and the polar equation for a conic section:

$$\frac{k}{m_1\ell^2} = \frac{1}{a(1-e^2)} \longrightarrow (1-e^2) = \frac{m_1\ell^2}{ka}$$

and above

Using: $A = \frac{1}{2}\ell\tau \qquad A = \pi a^2\sqrt{1-e^2}$

$$\frac{1}{2}\ell\tau = \pi a^2\sqrt{\frac{m_1\ell^2}{ka}} \qquad \frac{1}{4}\cancel{\ell^2}\tau^2 = \pi^2 a^{\cancel{4}\,3} m_1 \cancel{\ell^2}/k\cancel{a}$$

$$\tau^2 = \frac{4\pi^2 a^3 m_1}{k} \qquad \frac{\tau^2}{a^3} = \frac{4\pi^2 m_1}{k} \qquad k = G(m_1+m_2)m_1$$

$$\boxed{\frac{\tau^2}{a^3} = \frac{4\pi^2}{G(m_1+m_2)}}$$