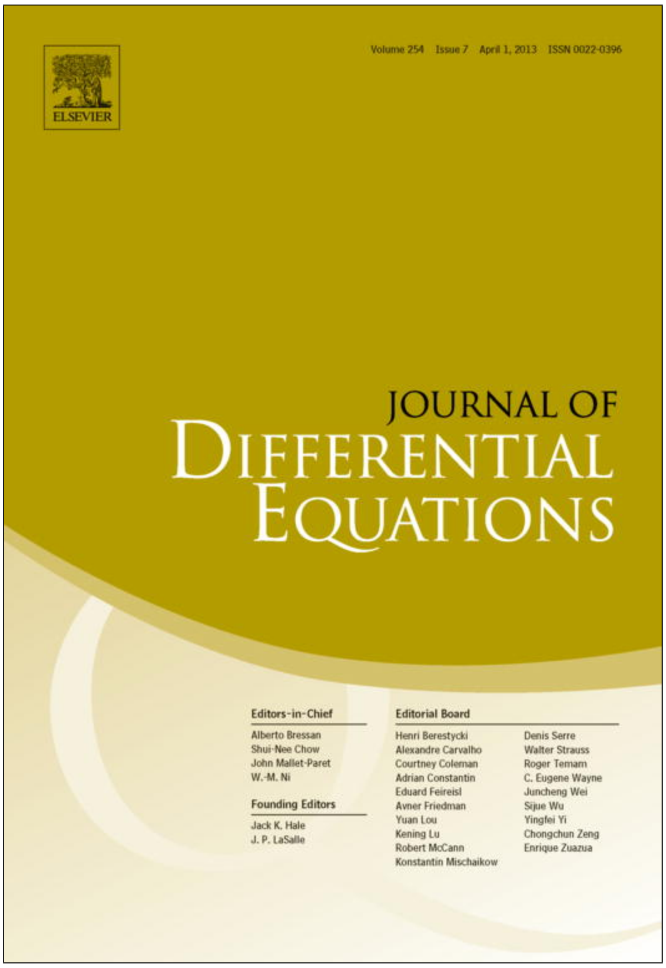
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The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion

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A B S T R A C T

This paper studies the global regularity of classical solutions to the 2D incompressible magnetohydrodynamic (MHD) equations with horizontal dissipation and horizontal magnetic diffusion. It is shown here that the horizontal component of any solution admits a global (in time) bound in any Lebesgue space L^{2r} with $1 \leqslant r < \infty$ and the bound grows no faster than the order of $\sqrt{r \log r}$ as r increases. In addition, we establish a conditional global regularity in terms of the $L^2_t L^\infty_x$-norm of the horizontal component and the global regularity of a slightly regularized version of the aforementioned MHD equations.

1. Introduction

The MHD equations govern the dynamics of the velocity and the magnetic field in electrically conducting fluids such as plasmas and reflect the basic physics conservation laws. They have been at the center of numerous analytical, experimental, and numerical investigations. One of the most fundamental problems concerning the MHD equations is whether their classical solutions are globally regular for all time or they develop singularities. This problem can be extremely difficult due to the nonlinear coupling between the Navier–Stokes equations with a forcing induced by the magnetic field and the induction equation. The 2D incompressible MHD equations concerned here can be represented in the form

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$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \nu_1 \partial_x^2 u + \nu_2 \partial_y^2 u + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = \eta_1 \partial_x^2 b + \eta_2 \partial_y^2 b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.1}$$

where $(x, y) \in \mathbb{R}^2$, $t \geqslant 0$, $u = (u_1(x, y, t), u_2(x, y, t))$ denotes the 2D velocity field, $p = p(x, y, t)$ the pressure, $b = (b_1(x, y, t), b_2(x, y, t))$ the magnetic field, and ν_1, ν_2, η_1 and η_2 are nonnegative real parameters. When $\nu_1 = \nu_2$ and $\eta_1 = \eta_2$, (1.1) reduces to the standard incompressible MHD equations.

When all four parameters ν_1, ν_2, η_1 and η_2 are positive, it is not hard to show that (1.1) possesses a unique global solution corresponding to sufficiently smooth initial data (see, e.g., [3,4]). If all four parameters are zero, (1.1) becomes inviscid and the global regularity problem appears to be out of reach. The intermediate cases when some of the parameters are positive have recently attracted considerable attention (see, e.g., [1,5]). As far as we know, the only cases for which the global well-posedness is known is when $\nu_1 > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 > 0$ or when $\nu_1 = 0$, $\nu_2 > 0$, $\eta_1 > 0$ and $\eta_2 = 0$. The global regularity for these two cases was recently established by Cao and Wu [1]. A partial answer for the case when $\nu_1 = \nu_2 = 0$, $\eta_1 > 0$ and $\eta_2 > 0$ was obtained in [1] and [5]. The MHD equations in this case are shown to possess global H^1 weak solutions. However, the uniqueness of such weak solutions and a global H^2-bound remain unknown. Many attempts have also been made on the MHD equations with only dissipation, namely (1.1) with $\nu_1 > 0$, $\nu_2 > 0$, $\eta_1 = \eta_2 = 0$, but the global regularity problem for this case remains open.

This paper is devoted to the case when $\nu_1 > 0$, $\nu_2 = 0$, $\eta_1 > 0$ and $\eta_2 = 0$, namely the MHD equations with horizontal dissipation and horizontal magnetic diffusion

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p + \partial_x^2 u + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b = \partial_x^2 b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \end{cases} \tag{1.2}$$

where we have set $\nu_1 = \eta_1 = 1$. We do not have a complete solution to the global regularity problem in this case. This paper presents several global *a priori* bounds and conditioned global regularity. In addition, we obtain the global regularity for a slightly regularized version of (1.2), namely

$$\begin{cases} \partial_t u + u \cdot \nabla u + \epsilon(-\Delta)^\delta u = -\nabla p + \partial_x^2 u + b \cdot \nabla b, \\ \partial_t b + u \cdot \nabla b + \epsilon(-\Delta)^\delta b = \partial_x^2 b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0 \end{cases} \tag{1.3}$$

with $\epsilon > 0$ and $\delta > 0$. These results indicate that we are close to a resolution and therefore give us confidence to predict the global regularity of (1.2).

Let us first try to understand the difficulty we would encounter when the energy method is applied. For any given sufficiently smooth datum

$$u(x, y, 0) = u_0(x, y), \qquad b(x, y, 0) = b_0(x, y),$$

say, $(u_0, b_0) \in H^2(\mathbb{R}^2)$, the corresponding solution obviously obeys global L^2-bound. That is,

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\int_0^t \|\partial_x u(\tau)\|_2^2 \, d\tau + 2\int_0^t \|\partial_x b(\tau)\|_2^2 \, d\tau = \|u_0\|_2^2 + \|b_0\|_2^2, \tag{1.4}$$

where we have written $\|f\|_p$ to denote the L^p-norm of $f \in L^p(\mathbb{R}^2)$ with $1 \leqslant p \leqslant \infty$. The trouble arises when we try to obtain the global H^1-bound. If we resort to the equations for the vorticity $\omega = \nabla \times u$ and the current density $j = \nabla \times b$, namely

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = \partial_x^2 \omega + b \cdot \nabla j, \\ \partial_t j + u \cdot \nabla j = \partial_x^2 j + b \cdot \nabla \omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \end{cases} \tag{1.5}$$

we then obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\|\omega\|_2^2 + \|j\|_2^2\big) + \|\partial_x \omega\|_2^2 + \|\partial_x j\|_2^2 \\ &\quad = 2\int j\big(\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1)\big)\,dx\,dy. \end{aligned} \tag{1.6}$$

In order to obtain suitable bounds for the terms on the right, we need the anisotropic Sobolev inequalities stated in the following lemma (see [1]).

Lemma 1.1. *If $f, g, h, \partial_y g, \partial_x h \in L^2(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} |fgh|\,dx\,dy \leqslant C\|f\|_2\|g\|_2^{\frac{1}{2}}\|\partial_y g\|_2^{\frac{1}{2}}\|h\|_2^{\frac{1}{2}}\|\partial_x h\|_2^{\frac{1}{2}}, \tag{1.7}$$

where C is a constant.

If we apply (1.7), two terms on the right of (1.6), $\int j\partial_x b_1 \partial_x u_2$ and $\int j\partial_x u_1 \partial_x b_2$ can be bounded suitably. Unfortunately, we do not know how to bound the other two terms in order to close the inequality in (1.6). This is where the direct energy method breaks down.

Motivated by a recent work of Cao and Wu on the 2D Boussinesq equation with partial dissipation [2], we explore here how the Lebesgue norm of the horizontal component (u_1, b_1) of a solution would affect the global regularity. First, we are able to obtain a global *a priori* bound for the norm $\|(u_1, b_1)\|_{2r}$ with $1 \leqslant r < \infty$, where $\|f\|_q$ with $1 \leqslant q \leqslant \infty$ denotes the norm of a function f in the Lebesgue space L^q. The precise statement for this global bound is given in Theorem 2.1 of Section 2. The bound depends exponentially on r and we do not know whether or not $\|(u_1, b_1)\|_\infty$ can be bounded for all time. If we do know that

$$\int_0^T \big\|(u_1, b_1)\big\|_\infty^2\,dt < \infty, \tag{1.8}$$

then we can actually show that the solution is regular on $[0, T]$. This is the conditional global regularity result established in Theorem 5.1 of Section 5.

Our main efforts are devoted to improving the global bound for $\|(u_1, b_1)\|_{2r}$. We are able to show that $\|(u_1, b_1)\|_{2r} \leqslant C\sqrt{r\log r}$ for large $r < \infty$. More precisely, we have the following theorem.

Theorem 1.2. *Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u, b) be the corresponding solution of (1.2). Let $2 < r < \infty$. Then,*

$$\big\|(u_1, b_1)(t)\big\|_{2r} \leqslant B_0(t)\sqrt{r\log r} + B_1, \tag{1.9}$$

where B_0 is a smooth function of t and B_1 depends only on $\|(u_0, b_0)\|_{2r}$.

The proof of this theorem is presented in Section 4. It relies heavily on the global bounds on the pressure p. As a preparation for Theorem 1.2, we first prove in Section 3 that the pressure associated with any classical solution obeys the global bound, for any $T > 0$ and $t < T$,

$$\|p(\cdot,t)\|_q \leqslant C(T), \qquad \int_0^T \|p(\cdot,t)\|_{H^s}^2\,dt < C(T),$$

where $1 < q \leqslant 3$ and $0 < s < 1$. We defy the details to Theorem 3.1 in Section 3. We are unable to prove a global bound for the case when $s = 1$. These global bounds together with a decomposition of the pressure into low and high frequency parts eventually lead to the global bound in Theorem 1.2.

The proofs of our results take advantage of the symmetric structure of (1.2). That is,

$$w^{\pm} = u \pm b$$

satisfies

$$\begin{cases} \partial_t w^+ + (w^- \cdot \nabla) w^+ = -\nabla p + \partial_x^2 w^+, \\ \partial_t w^- + (w^+ \cdot \nabla) w^- = -\nabla p + \partial_x^2 w^-, \\ \nabla \cdot w^+ = 0, \quad \nabla \cdot w^- = 0. \end{cases} \tag{1.10}$$

We remark that even this symmetric formulation is still more complex than the 2D Boussinesq equations dealt with in [2]. (1.10) consists of a system of two vector equations and the interaction between them makes it more difficult mathematically. Due to the lack of the global bound for $\int_0^t \|p(\tau)\|_{H^1}\,d\tau$, the proof of Theorem 1.2 does not directly follow from the methods in [2] on the 2D Boussinesq equations. New tools such as the triple product estimate involving fractional derivatives (see Lemma 4.1) are needed to cope with the more difficult situation here.

Finally we outline the plan for the rest of this paper. Section 2 presents a global bound for $\|(u_1,b_1)\|_{2r}$ while Section 3 proves the global bounds for the pressure as well as several other global bounds. Section 4 is mainly devoted to proving Theorem 1.2 and, as a preparation, an estimate for a triple product involving fractional derivatives is provided. Section 5 proves the conditional global regularity under (1.8). The last section shows that the slightly regularized system (1.3) always possesses global classical solutions.

2. A global bound in the Lebesgue spaces

Assume that (u,b) is a classical solution of (1.2). This section shows that its component in the x-direction (u_1,b_1) admits a global (in time) bound in $L^{2r}(\mathbb{R}^2)$ for any $1 \leqslant r < \infty$. The bound obtained here depends exponentially on r. More precisely, we have the following theorem.

Theorem 2.1. *Assume that $(u_0,b_0) \in H^2(\mathbb{R}^2)$ and let (u,b) be the corresponding solution of (1.2). Then, for any $1 \leqslant r < \infty$, (u_1,b_1) obeys the global bound*

$$\|(u_1,b_1)\|_{2r} \leqslant C_1 e^{C_2 r^3}, \tag{2.1}$$

where C_1 and C_2 are constants depending on $\|(u_0,b_0)\|_{2r}$ only.

In order to prove this theorem, we need the global L^2-bound.

Lemma 2.2. *Let $(u_0,b_0) \in H^2(\mathbb{R}^2)$ and let (u,b) be the corresponding solution of (1.2). Then, (u,b) obeys the following global L^2-bound,*

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\int_0^t \|\partial_x u(\tau)\|_2^2\,d\tau + 2\int_0^t \|\partial_x b(\tau)\|_2^2\,d\tau \leqslant \|u_0\|_2^2 + \|b_0\|_2^2 \tag{2.2}$$

for any $t \geqslant 0$.

Proof of Theorem 2.1. It is more convenient to use the symmetric form of (1.2), namely (1.10). Multiplying the first component of the first equation of (1.10) by $w_1^+|w_1^+|^{2r-2}$ and integrating with respect to space variable, we obtain, after integration by parts,

$$\frac{1}{2r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r} + (2r-1)\int |\partial_x w_1^+|^2 |w_1^+|^{2r-2} = (2r-1)\int p\partial_x w_1^+ |w_1^+|^{2r-2}. \tag{2.3}$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}\int p\partial_x w_1^+ |w_1^+|^{2r-2} &\leqslant \|p\|_{2r} \big\|\partial_x w_1^+ |w_1^+|^{r-1}\big\|_2 \big\||w_1^+|^{r-1}\big\|_{\frac{2r}{r-1}}\\ &\leqslant Cr\|\nabla p\|_{\frac{2r}{r+1}} \big\|\partial_x w_1^+ |w_1^+|^{r-1}\big\|_2 \|w_1^+\|_{2r}^{r-1},\end{aligned}$$

where C is a constant independent of r. Therefore, by Young's inequality,

$$(2r-1)\int p\partial_x w_1^+ |w_1^+|^{2r-2} \leqslant \frac{2r-1}{4}\big\|\partial_x w_1^+ |w_1^+|^{r-1}\big\|_2^2 + Cr^3\|\nabla p\|_{\frac{2r}{r+1}}^2 \|w_1^+\|_{2r}^{2(r-1)}.$$

To bound the pressure, we take the divergence of (1.10) to get

$$-\Delta p = \partial_x\big(w_1^-\partial_x w_1^+ + w_1^+\partial_x w_1^-\big) + \partial_y\big(w_1^+\partial_x w_2^- + w_1^-\partial_x w_2^+\big). \tag{2.4}$$

Due to the boundedness of Riesz transforms on L^q for any $1 < q < \infty$, we have

$$\begin{aligned}\|\nabla p\|_{\frac{2r}{r+1}} &\leqslant \big\|w_1^-\partial_x w_1^+\big\|_{\frac{2r}{r+1}} + \big\|w_1^+\partial_x w_1^-\big\|_{\frac{2r}{r+1}} + \big\|w_1^+\partial_x w_2^-\big\|_{\frac{2r}{r+1}} + \big\|w_1^-\partial_x w_2^+\big\|_{\frac{2r}{r+1}}\\ &\leqslant \|w_1^-\|_{2r}\big(\|\partial_x w_1^+\|_2 + \|\partial_x w_2^+\|_2\big) + \|w_1^+\|_{2r}\big(\|\partial_x w_1^-\|_2 + \|\partial_x w_2^-\|_2\big).\end{aligned} \tag{2.5}$$

Consequently,

$$\begin{aligned}&Cr^3\|\nabla p\|_{\frac{2r}{r+1}}^2 \|w_1^+\|_{2r}^{2(r-1)}\\ &\quad\leqslant Cr^3\big(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2\big)\big(\|w_1^-\|_{2r} + \|w_1^+\|_{2r}\big)^2 \|w_1^+\|_{2r}^{2(r-1)}\\ &\quad\leqslant Cr^3\big(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2\big)\big(\|w_1^+\|_{2r}^{2r} + \|w_1^-\|_{2r}^{2r}\big).\end{aligned}$$

Combining the estimates above, we obtain

$$\begin{aligned}&\frac{1}{r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r} + \frac{(2r-1)}{2}\int |\partial_x w_1^+|^2 |w_1^+|^{2r-2}\\ &\quad\leqslant Cr^3\big(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2\big)\big(\|w_1^+\|_{2r}^{2r} + \|w_1^-\|_{2r}^{2r}\big).\end{aligned}$$

Similarly,

$$\frac{1}{r}\frac{d}{dt}\|w_1^-\|_{2r}^{2r}+\frac{(2r-1)}{2}\int |\partial_x w_1^-|^2|w_1^-|^{2r-2}$$
$$\leqslant Cr^3\big(\|\partial_x w_1^+\|_2^2+\|\partial_x w_2^+\|_2^2+\|\partial_x w_1^-\|_2^2+\|\partial_x w_2^-\|_2^2\big)\big(\|w_1^+\|_{2r}^{2r}+\|w_1^-\|_{2r}^{2r}\big).$$

Adding these two inequalities yields

$$\frac{1}{r}\frac{d}{dt}\big(\|w_1^+\|_{2r}^{2r}+\|w_1^-\|_{2r}^{2r}\big)+\frac{(2r-1)}{2}\int \big(|\partial_x w_1^+|^2|w_1^+|^{2r-2}+|\partial_x w_1^-|^2|w_1^-|^{2r-2}\big)$$
$$\leqslant Cr^3\big(\|\partial_x w_1^+\|_2^2+\|\partial_x w_2^+\|_2^2+\|\partial_x w_1^-\|_2^2+\|\partial_x w_2^-\|_2^2\big)\big(\|w_1^+\|_{2r}^{2r}+\|w_1^-\|_{2r}^{2r}\big).$$

It then follows from Gronwall's inequality that

$$\|w_1^+\|_{2r}^{2r}+\|w_1^-\|_{2r}^{2r}\leqslant \big(\|w_1^+(0)\|_{2r}^{2r}+\|w_1^-(0)\|_{2r}^{2r}\big)$$
$$\times \exp\left(Cr^4\int_0^t \big(\|\partial_x w_1^+\|_2^2+\|\partial_x w_2^+\|_2^2+\|\partial_x w_1^-\|_2^2+\|\partial_x w_2^-\|_2^2\big)\,d\tau\right).$$

This inequality together with (2.2) then yields (2.1). □

3. Global bounds for the pressure

Several more global *a priori* bounds are established in this section. First we provide the L^{2r}-bounds for the second components (u_2, b_2) when $r = 2$ and 3. Then we establish two global bounds for the pressure: one for $\|p\|_q$ with $1 < q \leqslant 3$ and one for $\int_0^t \|p(\tau)\|_{H^s}^2\,d\tau$ with $s \in [0, 1)$. The precise results can be stated as follows.

Theorem 3.1. *Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u, b) be the corresponding solution of (1.2). Let p be the corresponding pressure. Let $s \in (0, 1)$. Then, for any $T > 0$ and $t \leqslant T$,*

$$\big\|(u_2, b_2)(t)\big\|_{2r}\leqslant C,\quad r=2,3, \tag{3.1}$$

and, for any $1 < q \leqslant 3$,

$$\big\|p(t)\big\|_q\leqslant C,\qquad \int_0^T \big\|p(\tau)\big\|_{H^s}^2\,d\tau < C, \tag{3.2}$$

where C is a constant depending on T and the initial data.

Proof. There does not appear to be a uniform approach to prove the bounds in (3.1) simultaneously for $r = 2$ and $r = 3$. We prove them separately and start with the L^4-bound. It is more convenient to use the symmetric form (1.10). Multiplying the second component of the first equation of (1.10) by $w_2^+|w_2^+|^2$ and integrating by parts yield

$$\frac{1}{4}\frac{d}{dt}\|w_2^+\|_4^4+3\int |\partial_x w_2^+|^2|w_2^+|^2=3\int p\,\partial_y w_2^+|w_2^+|^2.$$

To bound the term on the right, we use $\nabla\cdot w^+ = 0$ and integrate by parts to get

$$\begin{aligned}\int p\partial_y w_2^+|w_2^+|^2 &= -\int p\partial_x w_1^+|w_2^+|^2\\ &= \int \partial_x p w_1^+|w_2^+|^2 + 2\int p w_1^+\partial_x w_2^+ w_2^+\\ &= J_1 + 2J_2. \end{aligned}\tag{3.3}$$

By Hölder's and Sobolev's inequalities,

$$\begin{aligned}|J_2| &\leqslant \|p\|_4\|w_1^+\|_4\|w_2^+\partial_x w_2^+\|_2\\ &\leqslant C\|\nabla p\|_{\frac{4}{3}}\|w_1^+\|_4\|w_2^+\partial_x w_2^+\|_2.\end{aligned}$$

According to (2.5),

$$\|\nabla p\|_{\frac{4}{3}} \leqslant \|w_1^-\|_4\big(\|\partial_x w_1^+\|_2 + \|\partial_x w_2^+\|_2\big) + \|w_1^+\|_4\big(\|\partial_x w_1^-\|_2 + \|\partial_x w_2^-\|_2\big).$$

Therefore, by Young's inequality,

$$\begin{aligned}|J_2| &\leqslant \frac{1}{16}\|w_2^+\partial_x w_2^+\|_2^2\\ &\quad + C\big(\|w_1^-\|_4^4 + \|w_1^+\|_4^4\big)\big(\|\partial_x w_1^+\|_2^2 + \|\partial_x w_2^+\|_2^2 + \|\partial_x w_1^-\|_2^2 + \|\partial_x w_2^-\|_2^2\big).\end{aligned}$$

To bound J_1, we first apply Hölder's inequality,

$$|J_1| \leqslant \|\partial_x p\|_{\frac{8}{5}}\|w_1^+\|_8\|(w_2^+)^2\|_4.$$

By Lemma 3.2 below and $\nabla\cdot w^+ = 0$,

$$\|(w_2^+)^2\|_4 \leqslant C\|\partial_x(w_2^+)^2\|_2^{\frac{1}{2}}\|\partial_y(w_2^+)^2\|_1^{\frac{1}{2}} \leqslant C\|w_2^+\partial_x w_2^+\|_2^{\frac{1}{2}}\|w_2^+\partial_x w_1^+\|_1^{\frac{1}{2}}.$$

According to (2.5),

$$\begin{aligned}\|\nabla p\|_{\frac{8}{5}} &\leqslant C\|w_1^-\|_8\big(\|\partial_x w_1^+\|_2 + \|\partial_x w_2^+\|_2\big) + \|w_1^+\|_8\big(\|\partial_x w_1^-\|_2 + \|\partial_x w_2^-\|_2\big)\\ &\leqslant C\big(\|w_1^-\|_8 + \|w_1^+\|_8\big)\big(\|\partial_x w^-\|_2 + \|\partial_x w^+\|_2\big).\end{aligned}$$

Therefore,

$$\begin{aligned}|J_1| &\leqslant C\|w_1^+\|_8\big(\|w_1^-\|_8 + \|w_1^+\|_8\big)\big(\|\partial_x w^-\|_2 + \|\partial_x w^+\|_2\big)\|w_2^+\partial_x w_2^+\|_2^{\frac{1}{2}}\|w_2^+\partial_x w_1^+\|_1^{\frac{1}{2}}\\ &\leqslant \frac{1}{16}\|w_2^+\partial_x w_2^+\|_2^2\\ &\quad + C\big(\|\partial_x w^-\|_2^2 + \|\partial_x w^+\|_2^2\big) + \|w_1^+\|_8^4\big(\|w_1^-\|_8 + \|w_1^+\|_8\big)^4\|w_2^+\|_2^2\|\partial_x w_1^+\|_2^2.\end{aligned}$$

Inserting the estimates for J_1 and J_2 in (3.3) and recalling Theorem 2.1, we obtain a global bound for $\|w_2^+\|_4$. The bound for $\|w_2{}^-\|_4 < C$ can be similarly established.

To prove the L^6-bound in (3.1), we obtain from (1.10) that

$$\begin{aligned}
&\frac{1}{6}\frac{d}{dt}\left(\|w_2^+\|_6^6+\|w_2^-\|_6^6\right)+5\left\||w_2^+|^2|\partial_x w_2^+|\right\|_2^2+5\left\||w_2^-|^2|\partial_x w_2^-|\right\|_2^2\\
&\quad=5\int p\left(|w_2^+|^4\partial_y w_2^+ +|w_2^-|^4\partial_y w_2^-\right)\\
&\quad=-5\int p\left(|w_2^+|^4\partial_x w_1^+ +|w_2^-|^4\partial_x w_1^-\right)\\
&\quad=5\int \partial_x p\left(|w_2^+|^4 w_1^+ +|w_2^-|^4 w_1^-\right)+20\int p\left(|w_2^+|^3\partial_x w_2^+ w_1^+ +|w_2^-|^3\partial_x w_2^- w_1^-\right).
\end{aligned}$$

Applying Hölder's inequality, (2.4) and Lemma 3.2, we have

$$\begin{aligned}
&\int \partial_x p\left(|w_2^+|^4 w_1^+ +|w_2^-|^4 w_1^-\right)\\
&\quad\leqslant \|\partial_x p\|_{\frac{36}{19}}\left(\left\||w_2^+|^3\right\|_3^{4/3}\|w_1^+\|_{36}+\left\||w_2^-|^3\right\|_3^{4/3}\|w_1^-\|_{36}\right)\\
&\quad\leqslant C\left(\|w_1^+\|_{36}+\|w_1^-\|_{36}\right)^2\left(\|\partial_x w^+\|_2+\|\partial_x w^-\|_2\right)\\
&\qquad\times\left(\left\|\partial_y|w_2^+|^3\right\|_1^{\frac{4}{9}}\left\|\partial_x|w_2^+|^3\right\|_2^{\frac{4}{9}}\left\||w_2^+|^3\right\|_2^{\frac{4}{9}}+\left\|\partial_y|w_2^-|^3\right\|_1^{\frac{4}{9}}\left\|\partial_x|w_2^-|^3\right\|_2^{\frac{4}{9}}\left\||w_2^-|^3\right\|_2^{\frac{4}{9}}\right)\\
&\quad\leqslant C\left(\|w_1^+\|_{36}+\|w_1^-\|_{36}\right)^2\left(\|\partial_x w^+\|_2+\|\partial_x w^-\|_2\right)\left(\left\||w_2^+|^3\right\|_2^{\frac{4}{9}}+\left\||w_2^-|^3\right\|_2^{\frac{4}{9}}\right)\\
&\qquad\times\left(\|w_2^+\|_4^{\frac{8}{9}}\|\partial_y w_2^+\|_2^{\frac{4}{9}}+\|w_2^-\|_4^{\frac{8}{9}}\|\partial_y w_2^-\|_2^{\frac{4}{9}}\right)\left(\left\|\partial_x|w_2^+|^3\right\|_2^{\frac{4}{9}}+\left\|\partial_x|w_2^-|^3\right\|_2^{\frac{4}{9}}\right).
\end{aligned}$$

Also, by Hölder's inequality and (2.4),

$$\begin{aligned}
&\int p\left(|w_2^+|^3\partial_x w_2^+ w_1^+ +|w_2^-|^3\partial_x w_2^- w_1^-\right)\\
&\quad\leqslant \|p\|_6\left(\|w_2^+\|_6\left\||w_2^+|^2\partial_x w_2^+\right\|_2\|w_1^+\|_6+\|w_2^-\|_6\left\||w_2^-|^2\partial_x w_2^-\right\|_2\|w_1^-\|_6\right)\\
&\quad\leqslant C\left(\|w_1^+\|_6+\|w_1^-\|_6\right)\left(\|\partial_x w^+\|_2+\|\partial_x w^-\|_2\right)\\
&\qquad\times\left(\|w_1^+\|_6\left\||w_2^+|^2\partial_x w_2^+\right\|_2\|w_2^+\|_6+\|w_1^-\|_6\left\||w_2^-|^2\partial_x w_2^-\right\|_2\|w_2^-\|_6\right).
\end{aligned}$$

Therefore, by Young's and Gronwall's inequalities,

$$\|w_2^+\|_6^6+\|w_2^-\|_6^6+\int_0^t\left(\left\||w_2^+|^2|\partial_x w_2^+|\right\|_2^2+\left\||w_2^-|^2|\partial_x w_2^-|\right\|_2^2\right)\leqslant C.$$

We now prove the first inequality in (3.2). Taking the divergence of the first two equations in (1.10), we have

$$-\Delta p=\nabla\cdot\left(w^-\cdot\nabla w^+\right).$$

By the boundedness of Riesz transforms on L^q,

$$\|p\|_q\leqslant C\|w^-\|_{2q}\|w^+\|_{2q}.$$

For $1 < q \leqslant 3$, $\|w^-\|_{2q}$ and $\|w^+\|_{2q}$ are bounded according to Theorem 2.1 and (3.1) and thus $\|p\|_q < C$.

Now we prove the second inequality in (3.2). Recall that the operator Λ^s is defined through its Fourier transform, namely

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \hat{f}(\xi).$$

Combining (2.4), the boundedness of Riesz transforms on L^2 and the Hardy–Littlewood–Sobolev inequality, we have

$$\begin{aligned}
\|\Lambda^s p\|_2 &\leqslant \|\Lambda^s(-\Delta)^{-1}\partial_x(w_1^-\partial_x w_1^+ + w_1^+\partial_x w_1^-)\|_2 + \|\Lambda^s(-\Delta)^{-1}\partial_y(w_1^+\partial_x w_2^- + w_1^-\partial_x w_2^+)\|_2 \\
&\leqslant \|\Lambda^{-(1-s)}(w_1^-\partial_x w_1^+ + w_1^+\partial_x w_1^-)\|_2 + \|\Lambda^{-(1-s)}(w_1^+\partial_x w_2^- + w_1^-\partial_x w_2^+)\|_2 \\
&\leqslant C\|w_1^-\partial_x w_1^+ + w_1^+\partial_x w_1^-\|_q + \|w_1^+\partial_x w_2^- + w_1^-\partial_x w_2^+\|_q \\
&\leqslant C(\|\partial_x w^+\|_2 + \|\partial_x w^-\|_2)(\|w_1^+\|_{\frac{2}{1-s}} + \|w_1^-\|_{\frac{2}{1-s}}),
\end{aligned}$$

where q satisfies $\frac{1}{q} = \frac{1}{2} + \frac{1-s}{2}$ and C is a constant independent of s. This completes the proof of Theorem 3.1. $\square$

We have used two calculus inequalities of the following lemma.

Lemma 3.2. *Assume that $f \in L^2(\mathbb{R}^2)$, $\partial_x f \in L^1(\mathbb{R}^2)$ and $\partial_y f \in L^2(\mathbb{R}^2)$. Then*

$$\|f\|_4 \leqslant \sqrt{3}\|\partial_x f\|_1^{\frac{1}{2}}\|\partial_y f\|_2^{\frac{1}{2}}, \tag{3.4}$$

$$\|f\|_3 \leqslant \sqrt[3]{2}\|f\|_2^{\frac{1}{3}}\|\partial_x f\|_1^{\frac{1}{3}}\|\partial_y f\|_2^{\frac{1}{3}}. \tag{3.5}$$

Proof. We prove (3.4) and the proof of (3.5) is similar. Writing

$$f^4(x,y) = f^3(x,y)f(x,y) = \int_{-\infty}^{x} \partial_z(f^3(z,y))\,dz \int_{-\infty}^{y} \partial_z f(x,z)\,dz,$$

integrating in $(x,y) \in \mathbb{R}^2$ and applying Hölder's inequality yield (3.4). $\square$

4. An improved global Lebesgue bound

This section establishes the improved global bound for $\|(u_1,b_1)\|_{L^{2r}}$, which states that $\|(u_1,b_1)\|_{L^{2r}}$ grows at most at the order of $\sqrt{r\log r}$. We have already stated the precise result in Theorem 1.2 in the Introduction and we now prove it.

In order to prove this theorem, we need several facts that we now state and prove.

Lemma 4.1. *Let $q \in [2,\infty)$ and $s \in (\frac{1}{2},1]$. Assume that $f, g, \partial_y g \in L^2(\mathbb{R}^2)$, $h \in L^{2(q-1)}(\mathbb{R}^2)$ and $\Lambda_x^s h \in L^2(\mathbb{R}^2)$. Then,*

$$\left|\iint_{R^2} fgh\,dx\,dy\right| \leqslant C\|f\|_2\|g\|_2^{\rho}\|\partial_y g\|_2^{1-\rho}\|h\|_{2(q-1)}^{\vartheta}\|\Lambda_x^s h\|_2^{1-\vartheta}, \tag{4.1}$$

where ρ and ϑ are given by

$$\rho = \frac{1}{2} + \frac{(2s-1)(q-2)}{2(2s-1)(q-1)+2}, \qquad \vartheta = \frac{(2s-1)(q-1)}{(2s-1)(q-1)+1},$$

and Λ_x^s denotes a fractional derivative with respect to x and is defined by

$$\Lambda_x^s h(x) = \int e^{ix\cdot\xi} |\xi_1|^s \hat{h}(\xi)\, d\xi.$$

Proof. To prove this inequality, we recall the one-dimensional Sobolev inequality

$$\|h\|_{L_x^\infty(\mathbb{R})} \leqslant C\, \|h\|^{\vartheta}_{L_x^{2(q-1)}(\mathbb{R})} \big\| \Lambda_x^s h \big\|^{1-\vartheta}_{L_x^2(\mathbb{R})}, \tag{4.2}$$

where we have used the sub-index x with the Lebesgue spaces to emphasize that the norms are taken in one-dimensional Lebesgue spaces with respect to x. By Hölder's inequality and (4.2),

$$\begin{aligned} \left| \iint fgh\, dx\, dy \right| &\leqslant C \int \|f\|_{L_x^2} \|g\|_{L_x^2} \|h\|^{\vartheta}_{L_x^{2(q-1)}} \big\| \Lambda_x^s h \big\|^{1-\vartheta}_{L_x^2}\, dy \\ &\leqslant C \left(\int \|f\|^2_{L_x^2}\, dy \right)^{\frac{1}{2}} \left(\int \|g\|^{\mu}_{L_x^2}\, dy \right)^{\frac{1}{\mu}} \\ &\quad \times \left(\int \|h\|^{2(q-1)}_{L_x^{2(q-1)}}\, dy \right)^{\frac{\vartheta}{2(q-1)}} \left(\int \big\| \Lambda_x^s h \big\|^2_{L_x^2}\, dy \right)^{\frac{(1-\vartheta)}{2}} \\ &= C \|f\|_2 \|g\|_{L_x^2 L_y^\mu} \|h\|^{\vartheta}_{2(q-1)} \big\| \Lambda_x^s h \big\|^{1-\vartheta}_2, \end{aligned} \tag{4.3}$$

where $\mu = 2(q-1)/(\vartheta(q-2))$. Clearly, $\mu \geqslant 2$. By Minkowski's inequality followed by a Sobolev inequality,

$$\|g\|_{L_x^2 L_y^\mu} \leqslant \|g\|_{L_y^\mu L_x^2} \leqslant C \|g\|_2^{\rho} \|\partial_y g\|_2^{1-\rho}. \tag{4.4}$$

Inserting (4.4) in (4.3) then yields the desired inequality in (4.1). □

The following lemma allows us to bound the high frequency and low frequency parts of a function in H^s $(0 < s < 1)$ separately.

Lemma 4.2. *Let $f \in H^s(\mathbb{R}^2)$ with $s \in (0,1)$. Let $R \in (0, \infty)$. Denote by $B(0,R)$ the ball centered at zero with radius R and by $\chi_{B(0,R)}$ the characteristic function on $B(0,R)$. Write*

$$f = \bar{f} + \tilde{f} \quad \text{with } \bar{f} = \mathcal{F}^{-1}(\chi_{B(0,R)} \mathcal{F} f) \text{ and } \tilde{f} = \mathcal{F}^{-1}\big((1-\chi_{B(0,R)})\mathcal{F} f\big), \tag{4.5}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and the inverse Fourier transform, respectively. Then we have the following estimates for $\bar{f}$ and $\tilde{f}$.

(1) *For a pure constant C_0 (independent of s),*

$$\|\bar{f}\|_\infty \leqslant \frac{C_0}{\sqrt{1-s}} R^{1-s} \|f\|_{H^s(\mathbb{R}^2)}. \tag{4.6}$$

(2) *For any $2 \leqslant q < \infty$ satisfying $1 - s - \frac{2}{q} < 0$, there is a constant C_1 independent of s, q, R and f such that*

$$\|\tilde{f}\|_q \leqslant C_1 q R^{1-s-\frac{2}{q}} \|f\|_{H^s(\mathbb{R}^2)}. \tag{4.7}$$

This lemma complements Lemma 2.3 in [2]. The lemma there involves H^1-functions while this lemma allows to deal with H^s-functions with $s \in (0,1)$. This lemma can be similarly proven as Lemma 2.3 of [2], so the details are omitted.

We also recall Lemma 2.4 of [2].

Lemma 4.3. *Let $1 < q < \infty$. Let $f \in L^q(\mathbb{R}^d)$ and let $\tilde{f}$ be defined as in (4.5). Then, there exists a constant C depending on q only such that*

$$\|\tilde{f}\|_q \leqslant C\|f\|_q.$$

We are now ready to prove the main theorem of this section.

Proof of Theorem 1.2. As in the proof of Theorem 2.1, we use the symmetric form (1.10) and start with (2.3) with $r > 2$, namely

$$\frac{1}{2r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r} + (2r-1)\int \left|\partial_x w_1^+\right|^2 \left|w_1^+\right|^{2r-2} = (2r-1)\int p\,\partial_x w_1^+ \left|w_1^+\right|^{2r-2}. \tag{4.8}$$

The term on the right will be treated differently. To start, we fix $R > 0$ (to be specified later) and write

$$(2r-1)\int p\,\partial_x w_1^+ \left|w_1^+\right|^{2r-2} = J_1 + J_2,$$

where

$$J_1 = (2r-1)\int \bar{p}\,\partial_x w_1^+ \left|w_1^+\right|^{2r-2}, \qquad J_2 = (2r-1)\int \tilde{p}\,\partial_x w_1^+ \left|w_1^+\right|^{2r-2}$$

with $\bar{p}$ and $\tilde{p}$ as defined in (4.5). To estimate J_1 and J_2, we choose two parameters s and q satisfying

$$\frac{\sqrt{5}-1}{2} < s < 1, \qquad 2 < q \leqslant \frac{5}{2}, \qquad \frac{3}{2} + \frac{1}{2(2s-1)} < q < 1 + \frac{1}{1-s}. \tag{4.9}$$

The technical constraints in (4.9) will become clear later. By Hölder's and Young's inequalities, we find

$$\begin{aligned}
|J_1| &\leqslant (2r-1)\|\bar{p}\|_\infty \left\|\left|w_1^+\right|^{r-1}\right\|_2 \left\|\partial_x w_1^+\left(w_1^+\right)^{r-1}\right\|_2 \\
&\leqslant (2r-1)\|\bar{p}\|_\infty^2 \left\|\left|w_1^+\right|^{r-1}\right\|_2^2 + \frac{2r-1}{4}\left\|\partial_x w_1^+\left(w_1^+\right)^{r-1}\right\|_2^2.
\end{aligned}$$

Assuming s and q satisfying (4.9) and applying Lemma 4.2, we have

$$\|\bar{p}\|_\infty \leqslant \frac{C_0}{\sqrt{1-s}} R^{1-s}\|p\|_{H^s}, \tag{4.10}$$

where C_0 is a constant independent of s. In the rest of the proof we pay special attention to whether a constant is bounded uniformly as $s \to 1^-$. By (4.10) and the interpolation inequality

$$\int \left(w_1^+\right)^{2r-2} \leqslant \left\|w_1^+\right\|_2^{\frac{2}{r-1}} \left\|w_1^+\right\|_{2r}^{\frac{2r^2-4r}{r-1}}, \tag{4.11}$$

we have

$$\begin{aligned} |J_1| &\leqslant \frac{2r-1}{4} \left\|\partial_x w_1^+ \left(w_1^+\right)^{r-1}\right\|_2^2 \\ &\quad + \frac{C_0^2}{1-s}(2r-1)R^{2(1-s)}\|p\|_{H^s}^2 \left\|w_1^+\right\|_2^{\frac{2}{r-1}} \left\|w_1^+\right\|_{2r}^{\frac{2r^2-4r}{r-1}}, \end{aligned} \tag{4.12}$$

where C_0 is independent of s. To bound J_2, we first apply Lemma 4.1 to obtain

$$|J_2| \leqslant C(2r-1)\left\|\partial_x w_1^+ \left|w_1^+\right|^{r-1}\right\|_2 \|\tilde{p}\|_{2(q-1)}^{\vartheta} \left\|\Lambda_y^s \tilde{p}\right\|_2^{1-\vartheta} \left\|\left|w_1^+\right|^{r-1}\right\|_2^{\rho} \left\|\partial_x \left(w_1^+\right)^{r-1}\right\|_2^{1-\rho}$$

where s and q satisfy (4.9), ϑ and ρ are given explicitly in terms of s and q,

$$\vartheta = \frac{(2s-1)(q-1)}{(2s-1)(q-1)+1}, \qquad \rho = \frac{1}{2} + \frac{(2s-1)(q-2)}{2[(2s-1)(q-1)+1]}, \tag{4.13}$$

and C is bounded uniformly as $s \to 1^-$. According to (4.11), we have

$$\left\|\left|w_1^+\right|^{r-1}\right\|_2^{\rho} \leqslant \left\|w_1^+\right\|_2^{\frac{\rho}{r-1}} \left\|w_1^+\right\|_{2r}^{\frac{\rho(r^2-2r)}{(r-1)}}.$$

By Hölder's inequality,

$$\begin{aligned} \left\|\partial_x \left(w_1^+\right)^{r-1}\right\|_2^{1-\rho} &= (r-1)^{1-\rho} \left(\int \left(\partial_x w_1^+\right)^2 \left(w_1^+\right)^{2(r-2)}\right)^{\frac{1}{2}(1-\rho)} \\ &= (r-1)^{1-\rho} \left(\int \left(\partial_x w_1^+\right)^{\frac{2}{r-1}} \left(\partial_x w_1^+\right)^{\frac{2(r-2)}{r-1}} \left(w_1^+\right)^{2(r-2)}\right)^{\frac{1}{2}(1-\rho)} \\ &= (r-1)^{1-\rho} \left\|\partial_x w_1^+\right\|_2^{\frac{1-\rho}{r-1}} \left(\int \left(w_1^+\right)^{2(r-1)} \left(\partial_x w_1^+\right)^2\right)^{\frac{(r-2)(1-\rho)}{2(r-1)}}. \end{aligned}$$

Therefore, by Young's inequality,

$$\begin{aligned} |J_2| &\leqslant C(2r-1)(r-1)^{1-\rho} \left\|\partial_x w_1^+\right\|_2^{\frac{1-\rho}{r-1}} \left\|w_1^+\right\|_2^{\frac{\rho}{r-1}} \left\|w_1^+\right\|_{2r}^{\frac{\rho(r^2-2r)}{r-1}} \\ &\quad \times \|\tilde{p}\|_{2(q-1)}^{\vartheta} \left\|\Lambda^s \tilde{p}\right\|_2^{1-\vartheta} \left(\int \left(\partial_x w_1^+\right)^2 \left(w_1^+\right)^{2r-2}\right)^{\frac{1}{2}+\frac{(r-2)(1-\rho)}{2(r-1)}} \\ &\leqslant \frac{2r-1}{4} \int \left(\partial_x w_1^+\right)^2 \left(w_1^+\right)^{2r-2} + C(2r-1)(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}} \left\|w_1^+\right\|_2^{\frac{2\rho}{\sigma}} \\ &\quad \times \left\|\partial_x w_1^+\right\|_2^{\frac{2(1-\rho)}{\sigma}} \left\|w_1^+\right\|_{2r}^{\frac{2\rho(r^2-2r)}{\sigma}} \|\tilde{p}\|_{2(q-1)}^{\frac{2\vartheta(r-1)}{\sigma}} \left\|\Lambda^s \tilde{p}\right\|_2^{\frac{2(1-\vartheta)(r-1)}{\sigma}}, \end{aligned} \tag{4.14}$$

where C is again bounded uniformly as $s \to 1^-$, and, for notational convenience, we have written

$$\sigma = (r-1) - (1-\rho)(r-2) = 1 + \rho r - 2\rho. \tag{4.15}$$

To further the estimate, we split $\|\tilde{p}\|_{2(q-1)}$ into two parts and bound one of them via Lemma 4.2. More precisely, we have, for any $0 \leqslant \beta \leqslant 1$,

$$\begin{aligned}
\|\tilde{p}\|_{2(q-1)} &= \|\tilde{p}\|_{2(q-1)}^{1-\beta} \|\tilde{p}\|_{2(q-1)}^{\beta} \\
&\leqslant C\|\tilde{p}\|_{2(q-1)}^{1-\beta} R^{(1-s-\frac{1}{q-1})\beta} \|p\|_{H^s}^{\beta} \\
&\leqslant C\|p\|_{2(q-1)}^{1-\beta} R^{(1-s-\frac{1}{q-1})\beta} \|p\|_{H^s}^{\beta},
\end{aligned} \tag{4.16}$$

where the last inequality follows from Lemma 4.3 and C is a constant independent of s. Due to the condition on s and q in (4.9), this bound allows us to generate $R^{(1-s-\frac{1}{q-1})\beta}$ with $(1-s-\frac{1}{q-1})\beta \leqslant 0$. Inserting (4.16) in (4.14) yields

$$\begin{aligned}
|J_2| \leqslant{}& \frac{2r-1}{4} \int (\partial_x w_1^+)^2 (w_1^+)^{2r-2} \\
&+ C(2r-1)(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}} R^{(1-s-\frac{1}{q-1})\beta \frac{2\vartheta(r-1)}{\sigma}} \|w_1^+\|_2^{\frac{2\rho}{\sigma}} \\
&\times \|\partial_x w_1^+\|_2^{\frac{2(1-\rho)}{\sigma}} \|w_1^+\|_{2r}^{\frac{2\rho(r^2-2r)}{\sigma}} \|p\|_{2(q-1)}^{(1-\beta)\frac{2\vartheta(r-1)}{\sigma}} \|p\|_{H^s}^{\beta\frac{2\vartheta(r-1)}{\sigma} + \frac{2(1-\vartheta)(r-1)}{\sigma}},
\end{aligned}$$

where, again, C is bounded uniformly as $s \to 1^-$. We choose β so that the sum of the powers of $\|\partial_x w_1^+\|_2$ and of $\|p\|_{H^s}$ is equal to 2, namely

$$\frac{2(1-\rho)}{\sigma} + \beta\frac{2\vartheta(r-1)}{\sigma} + \frac{2(1-\vartheta)(r-1)}{\sigma} = 2.$$

Recalling (4.13) and (4.15), we find that

$$\beta = \frac{(2s-1)(2q-3)-1}{(2q-2)(2s-1)}. \tag{4.17}$$

The condition in (4.9) guarantees that $0 < \beta \leqslant 1$. Then

$$\|\partial_x w_1^+\|_2^{\frac{2(1-\rho)}{\sigma}} \|p\|_{H^s}^{\beta\frac{2\vartheta(r-1)}{\sigma} + \frac{2(1-\vartheta)(r-1)}{\sigma}} \leqslant C\big(\|\partial_x w_1^+\|_2^2 + \|p\|_{H^s}^2\big).$$

Therefore, for β given by (4.17), we have

$$\begin{aligned}
|J_2| \leqslant{}& \frac{2r-1}{4} \int (\partial_x w_1^+)^2 (w_1^+)^{2r-2} \\
&+ C(2r-1)(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}} R^{(1-s-\frac{1}{q-1})\beta \frac{2\vartheta(r-1)}{\sigma}} \|w_1^+\|_2^{\frac{2\rho}{\sigma}} \\
&\times \|p\|_{2(q-1)}^{(1-\beta)\frac{2\vartheta(r-1)}{\sigma}} \big(\|\partial_x w_1^+\|_2^2 + \|p\|_{H^s}^2\big) \|w_1^+\|_{2r}^{\frac{2\rho(r^2-2r)}{\sigma}}.
\end{aligned} \tag{4.18}$$

Combining (4.8), (4.12) and (4.18), we obtain

$$\begin{aligned}
&\frac{1}{2r}\frac{d}{dt}\|w_1^+\|_{2r}^{2r}+\frac{2r-1}{4}\int|\partial_x w_1^+|^2|w_1^+|^{2r-2}\\
&\quad\leqslant\frac{C_0^2}{1-s}(2r-1)R^{2(1-s)}\|p\|_{H^s}^2\|w_1^+\|_2^{\frac{2}{r-1}}\|w_1^+\|_{2r}^{\frac{2r^2-4r}{r-1}}\\
&\qquad+C(2r-1)(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}}R^{(1-s-\frac{1}{q-1})\beta\frac{2\vartheta(r-1)}{\sigma}}\|w_1^+\|_2^{\frac{2\rho}{\sigma}}\\
&\qquad\times\|p\|_{2(q-1)}^{(1-\beta)\frac{2\vartheta(r-1)}{\sigma}}\big(\|\partial_x w_1^+\|_2^2+\|p\|_{H^s}^2\big)\|w_1^+\|_{2r}^{\frac{2\rho(r^2-2r)}{\sigma}}
\end{aligned}\tag{4.19}$$

where C_0 is independent of s and C is bounded uniformly as $s\to 1^-$. We now choose R so that

$$R^{2(1-s)}=(r-1)^{\frac{2(1-\rho)(r-1)}{\sigma}}R^{(1-s-\frac{1}{q-1})\beta\frac{2\vartheta(r-1)}{\sigma}}.$$

Solving this equation for R, we find

$$R^{2(1-s)}=(r-1)^{\frac{2(1-s)(1-\rho)(r-1)}{(1-s)\sigma+\beta\vartheta(s-1+\frac{1}{q-1})(r-1)}}.$$

We then use (4.13), (4.15) and (4.17) to simplify this index and obtain

$$\frac{2(1-s)(1-\rho)(r-1)}{(1-s)\sigma+\beta\vartheta(s-1+\frac{1}{q-1})(r-1)}=\frac{2(1-s)(q-1)}{q-2+(r-1)^{-1}(1-s)(q-1)}.$$

We denote this index by δ,

$$\delta\equiv\frac{2(1-s)(q-1)}{q-2+(r-1)^{-1}(1-s)(q-1)}\tag{4.20}$$

and therefore $R^{2(1-s)}=(r-1)^{\delta}$. Clearly, $\delta\to 0$ as $s\to 1$, and

$$\frac{1}{1-s}=\frac{q-1}{q-2}\left(2-\frac{\delta}{r-1}\right)\frac{1}{\delta}\leqslant\frac{2q-2}{q-2}\frac{1}{\delta}.$$

In addition, we notice that

$$\frac{2r^2-4r}{r-1}\leqslant 2r-2,\qquad \frac{2\rho(r^2-2r)}{\sigma}\leqslant 2r-2.$$

Without loss of generality, we assume $\|w_1^+\|_{2r}\geqslant 1$. It then follows from (4.19) that

$$\frac{d}{dt}\|w_1^+\|_{2r}^2\leqslant\frac{C}{\delta}B(t)(2r-1)(r-1)^{\delta},\tag{4.21}$$

where C is bounded uniformly as $\delta\to 0^+$, and

$$B(t)=\|p\|_{H^s}^2\|w_1^+\|_2^{\frac{2}{r-1}}+\|w_1^+\|_2^{\frac{2\rho}{\sigma}}\|p\|_{2(q-1)}^{(1-\beta)\frac{2\vartheta(r-1)}{\sigma}}\big(\|\partial_x w_1^+\|_2^2+\|p\|_{H^s}^2\big).$$

Since (4.21) holds for any $\delta>0$, we set

$$\delta=\frac{1}{\log(r-1)}$$

to obtain the optimal upper bound

$$\frac{d}{dt}\|w_1^+\|_{2r}^2 \leqslant CB(t)(2r-1)\log(r-1). \tag{4.22}$$

After δ is selected, we then choose s and q satisfying (4.9) to fulfill (4.20). Since we have chosen $2 < q \leqslant \frac{5}{2}$, we have $2 < 2(q-1) \leqslant 3$ and, according to Theorem 3.1, $B(t)$ is integrable on any time interval. We obtain (1.9) after integrating (4.22) in time. This completes the proof of Theorem 1.2. $\square$

5. Conditional global regularity

This section establishes the global bounds for $\|(u,b)\|_{H^2}$ in terms of the norms of the horizontal components u_1 and b_1 in $L_t^2 L_x^\infty$. More precisely, we have the following theorem.

Theorem 5.1. *Assume $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u,b) be the corresponding solution of (1.2). If*

$$\int_0^T \|(u_1, b_1)(t)\|_\infty^2 \, dt < \infty$$

for some $T > 0$, then $\|(u,b)\|_{H^2}$ is finite on $[0,T]$.

The proof of this theorem is divided into two major parts. The first part bounds the H^1-norm while the second bounds the H^2-norm.

5.1. H^1 in terms of $\|(u_1,b_1)\|_{L_t^2 L_x^\infty}$

This subsection proves the following proposition.

Proposition 5.2. *Assume $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u,b) be the corresponding solution of (1.2). Then, for any $T > 0$ and $t \leqslant T$,*

$$\|(u,b)(t)\|_{H^1} \leqslant C_1 e^{C_2 \int_0^t (\|u_1(\tau)\|_\infty^2 + \|b_1(\tau)\|_\infty^2)\, d\tau}, \tag{5.1}$$

where C_1 depends on T and the initial data only and C_2 is a pure constant.

Proof. Taking the inner product of the first equation of (1.10) with Δw^+ and integrating by parts, we find

$$\frac{1}{2}\frac{d}{dt}\|\nabla w^+\|_2^2 + \|\partial_x \nabla w^+\|_2^2 = I_1 + I_2 + I_3 + I_4 + I_5 + I_6,$$

where

$$I_1 = \int \partial_x w_1^- \partial_x w_2^+ \partial_x w_2^+, \qquad I_2 = \int \partial_x w_2^- \partial_y w_1^+ \partial_x w_1^+, \qquad I_3 = \int \partial_x w_2^- \partial_y w_2^+ \partial_x w_2^+,$$

$$I_4 = \int \partial_y w_1^- \partial_x w_1^+ \partial_y w_1^+, \qquad I_5 = \int \partial_y w_1^- \partial_x w_2^+ \partial_y w_2^+, \qquad I_6 = \int \partial_y w_2^- \partial_y w_1^+ \partial_y w_1^+.$$

The terms can be bounded as follows. By Lemma 1.1,

$$
\begin{aligned}
|I_1| &\leqslant C\|\partial_x w_1^-\|_2 \|\partial_x w_2^+\|_2^{\frac{1}{2}} \|\partial_{xx}^2 w_2\|_2^{\frac{1}{2}} \|\partial_x w_2^+\|_2^{\frac{1}{2}} \|\partial_{xy}^2 w_2^+\|_2^{\frac{1}{2}} \\
&\leqslant C\|\partial_x w_1^-\|_2 \|\partial_x w_2^+\|_2 \|\nabla \partial_x w_2^+\|_2 \\
&\leqslant \frac{1}{16}\|\nabla \partial_x w_2^+\|_2^2 + C\|\partial_x w_1^-\|_2^2 \|\nabla w_2^+\|_2^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
|I_2| &\leqslant \frac{1}{16}\|\nabla \partial_x w_1^+\|_2^2 + C\|\partial_x w_2^-\|_2^2 \|\nabla w_1^+\|_2^2, \\
|I_3| &\leqslant \frac{1}{16}\|\nabla \partial_x w_2^+\|_2^2 + C\|\partial_x w_2^-\|_2^2 \|\nabla w_2^+\|_2^2.
\end{aligned}
$$

Integrating by parts, we have

$$
I_4 = -\int \partial_{xy}^2 w_1^- w_1^+ \partial_y w_1^+ - \int \partial_y w_1^- w_1^+ \partial_{xy}^2 w_1^+.
$$

By Hölder's inequality,

$$
\begin{aligned}
I_4 &\leqslant 2\|w_1^+\|_\infty \|\nabla \partial_x w_1^-\|_2 \|\nabla w_1^-\|_2 \\
&\leqslant \frac{1}{16}\|\nabla \partial_x w_1^-\|_2^2 + C\|w_1^+\|_\infty^2 \|\nabla w_1^-\|_2^2.
\end{aligned}
$$

I_5 and I_6 admit similar bounds as I_4,

$$
\begin{aligned}
|I_5| &\leqslant \frac{1}{16}\|\nabla w_x^+\|_2^2 + C\|\partial_x w^+\|_2^2 \|\nabla w_1^-\|_2^2, \\
|I_6| &\leqslant \frac{1}{16}\|\nabla \partial_x w_1^+\|^2 + C\|w_1^-\|_\infty^2 \|\nabla w_1^+\|_2^2.
\end{aligned}
$$

Similar estimates can be obtained for ∇w^-. Combining them yields

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\nabla w^+\|_2^2 + \|\nabla w^-\|_2^2\right) + \left(\|\partial_x \nabla w^+\|_2^2 + \|\partial_x \nabla w^-\|_2^2\right) \\
&\quad \leqslant \left(\|\partial_x w^+\|_2^2 + \|\partial_x w^-\|_2^2 + \|w_1^-\|_\infty^2 + \|w_1^+\|_\infty^2\right)\left(\|\nabla w^+\|_2^2 + \|\nabla w^-\|_2^2\right).
\end{aligned}
$$

Gronwall's lemma then yields the desired L^2-bound for $(\nabla u, \nabla b)$. Combining with the global L^2-bound in Lemma 2.2 leads to (5.1). □

5.2. *Proof of Theorem 5.1*

With the global bounds for the H^1-norm at our disposal, the goal of this subsection is to complete the proof of Theorem 5.1.

Proof of Theorem 5.1. Taking the inner product of the first equation in (1.10) with $\Delta^2 w^+$ and integrating by parts, we find

$$
\frac{1}{2}\frac{d}{dt}\|\Delta w^+\|_2^2 + \|\partial_x \Delta w^+\|_2^2 = -\int \Delta\left(w^- \cdot \nabla w^+\right) \cdot \Delta w^+. \tag{5.2}
$$

In order to make use of the anisotropic dissipation, we need to decompose the nonlinear term into different parts that show explicit dependence on the horizontal and vertical derivatives. We write

$$\int \Delta\big(w^- \cdot \nabla w^+\big) \cdot \Delta w^+ = J_1 + J_2 + J_3,$$

where

$$J_1 = \int \big(\Delta w^- \cdot \nabla w^+\big) \cdot \Delta w^+, \qquad J_2 = 2\int \big(\partial_x w^- \cdot \nabla \partial_x w^+\big) \cdot \Delta w^+,$$
$$J_3 = 2\int \big(\partial_y w^- \cdot \nabla \partial_y w^+\big) \cdot \Delta w^+.$$

We further decompose J_1 into four terms, $J_1 = J_{11} + J_{12} + J_{13} + J_{14}$, where

$$J_{11} = \int \big(\Delta w_1^- \partial_x w_1^+\big) \Delta w_1^+, \qquad J_{12} = \int \big(\Delta w_1^- \partial_x w_2^+\big) \Delta w_2^+,$$
$$J_{13} = \int \big(\Delta w_2^- \partial_y w_1^+\big) \Delta w_1^+, \qquad J_{14} = \int \big(\Delta w_2^- \partial_y w_2^+\big) \Delta w_2^+.$$

It is clear that, after integration by parts and applying Hölder's inequality,

$$|J_{11}| \leqslant \frac{1}{16}\big(\|\Delta \partial_x w_1^+\|_2^2 + \|\Delta \partial_x w_1^-\|_2^2\big) + 4\|w_1^+\|_\infty^2\big(\|\Delta w_1^+\|_2^2 + \|\Delta w_1^-\|_2^2\big).$$

Similarly, after invoking $\partial_x w_1^+ + \partial_y w_2^+ = 0$,

$$|J_{14}| \leqslant \frac{1}{16}\big(\|\Delta \partial_x w_2^+\|_2^2 + \|\Delta \partial_x w_2^-\|_2^2\big) + 4\|w_1^+\|_\infty^2\big(\|\Delta w_2^+\|_2^2 + \|\Delta w_2^-\|_2^2\big).$$

To bound J_{12}, we apply Lemma 1.1 to obtain

$$\begin{aligned}
|J_{12}| &\leqslant C\|\partial_x w_2^+\|_2 \|\Delta w_1^-\|_2^{\frac{1}{2}} \|\Delta \partial_x w_1^-\|_2^{\frac{1}{2}} \|\Delta w_2^+\|_2^{\frac{1}{2}} \|\Delta \partial_y w_2^+\|_2^{\frac{1}{2}} \\
&= C\|\partial_x w_2^+\|_2 \|\Delta w_1^-\|_2^{\frac{1}{2}} \|\Delta \partial_x w_1^-\|_2^{\frac{1}{2}} \|\Delta w_2^+\|_2^{\frac{1}{2}} \|\Delta \partial_x w_1^+\|_2^{\frac{1}{2}} \\
&\leqslant \frac{1}{16}\big(\|\Delta \partial_x w_1^+\|_2^2 + \|\Delta \partial_x w_1^-\|_2^2\big) + C\|\partial_x w_2^+\|_2^2\big(\|\Delta w_2^+\|_2^2 + \|\Delta w_1^-\|_2^2\big).
\end{aligned}$$

To bound J_{13}, we need the H^1-bound from Proposition 5.2. By Lemma 1.1,

$$\begin{aligned}
|J_{13}| &\leqslant C\|\Delta w_1^+\|_2 \|\Delta w_2^-\|_2^{\frac{1}{2}} \|\Delta \partial_y w_2^-\|_2^{\frac{1}{2}} \|\partial_y w_1^+\|_2^{\frac{1}{2}} \|\partial_x \partial_y w_1^+\|_2^{\frac{1}{2}} \\
&\leqslant C\|\Delta w_1^+\|_2 \|\Delta w_2^-\|_2^{\frac{1}{2}} \|\Delta \partial_x w_1^-\|_2^{\frac{1}{2}} \|\nabla w_1^+\|_2^{\frac{1}{2}} \|\nabla \partial_x w_1^+\|_2^{\frac{1}{2}} \\
&\leqslant \frac{1}{16}\|\Delta \partial_x w_1^-\|_2^2 + C\|\nabla w_1^+\|_2 \|\Delta w_1^+\|_2^2 + C\|\partial_x \nabla w_1^+\|_2^2 \|\Delta w_2^-\|_2^2.
\end{aligned}$$

Collecting the estimates for J_1, we have found that four terms

$$
\begin{aligned}
|J_1| \leqslant{}& \frac{1}{4}\left(\|\Delta\partial_x w^+\|_2^2 + \|\Delta\partial_x w^-\|_2^2\right)\\
&+ C\left(\|w_1^+\|_\infty^2 + \|\partial_x w^+\|_2^2 + \|\nabla w_1^+\|_2 + \|\partial_x \nabla w_1^+\|_2^2\right)\left(\|\Delta w^+\|_2^2 + \|\Delta w^-\|_2^2\right).
\end{aligned}
$$

J_2 and J_3 can be estimated in a similar fashion and we omit further details. Similar estimates can be obtained for Δw^-. Combining the estimates for all of them and applying Gronwall's inequality then yield the desired global result. This completes the proof of Theorem 5.1. □

6. Global regularity for a slightly regularized system

This section establishes that (1.3) possesses global regular solutions if the initial data are sufficiently smooth. More precisely, we have the following theorem.

Theorem 6.1. *Let $\epsilon > 0$ and $\delta > 0$ be real parameters. Consider (1.3) with an initial data $(u_0, b_0) \in H^2(\mathbb{R}^2)$. Then the corresponding solution (u, b) obeys the following global a priori bounds, for any $T > 0$ and $t \leqslant T$,*

$$
\|(u,b)\|_{H^2}^2 + \int_0^t \left(\|(\partial_x u, \partial_x b)\|_{H^2}^2 + \epsilon\left\|\left(\Lambda^\delta u, \Lambda^\delta b\right)\right\|_{H^2}^2\right) d\tau \leqslant C,
$$

where C is a constant depending on T and $\|(u_0, b_0)\|_{H^2}$ only.

Proof. We show that (u, b) admits a global H^2-bound. Clearly,

$$
\begin{aligned}
&\|(u,b)(t)\|_2^2 + 2\int_0^t \left(\|\partial_x u(\tau)\|_2^2 + \|\partial_x b(\tau)\|_2^2\right) d\tau\\
&\quad + 2\epsilon \int_0^t \left(\|\Lambda^\delta u(\tau)\|_2^2 + \|\Lambda^\delta b(\tau)\|_2^2\right) d\tau = \|(u_0, b_0)\|_2^2.
\end{aligned}
$$

To obtain the global bound for the H^1-norm, we take advantage of the vorticity formulation. Taking the curl of (1.3), we find that $\omega = \nabla \times u$ and $j = \nabla \times b$ satisfy

$$
\begin{cases}
\partial_t \omega + u\cdot\nabla\omega + \epsilon(-\Delta)^\delta \omega = b\cdot\nabla j + \partial_x^2 \omega,\\
\partial_t j + u\cdot\nabla j + \epsilon(-\Delta)^\delta j = b\cdot\nabla\omega + \partial_x^2 j\\
\quad + 2\partial_x b_1(\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1(\partial_y b_1 + \partial_x b_2).
\end{cases}
\tag{6.1}
$$

Taking the inner product of (6.1) with (ω, j) and integrating by parts, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_2^2 + \|j\|_2^2\right) + \|\partial_x\omega\|_2^2 + \|\partial_x j\|_2^2 + \epsilon\|\Lambda^\delta\omega\|_2^2 + \epsilon\|\Lambda^\delta j\|_2^2\\
&\quad = J_1 + J_2 + J_3 + J_4,
\end{aligned}
\tag{6.2}
$$

where

$$
\begin{aligned}
J_1 &= 2\int \partial_x b_1 \partial_y u_1 j\,dx\,dy, \qquad J_2 = 2\int \partial_x b_1 \partial_x u_2 j\,dx\,dy,\\
J_3 &= 2\int \partial_x u_1 \partial_y b_1 j\,dx\,dy, \qquad J_4 = 2\int \partial_x u_1 \partial_x b_2 j\,dx\,dy.
\end{aligned}
$$

The terms above can be bounded as follows. Integrating by parts, we have

$$J_1 = -2\int b_1 \partial_{xy} u_1 j - 2\int b_1 \partial_y u_1 \partial_x j.$$

Choose q large enough such that $q\delta > 2$. By Hölder's inequality,

$$|J_1| \leqslant 2\|b_1\|_q \|\partial_{xy} u_1\|_2 \|j\|_{\frac{2q}{q-2}} + 2\|b_1\|_q \|\partial_x j\|_2 \|\partial_y u_1\|_{\frac{2q}{q-2}}. \tag{6.3}$$

By the boundedness of singular integral operators,

$$\|\partial_{xy} u_1\|_2 \leqslant C\|\partial_x \omega\|_2, \qquad \|\partial_y u_1\|_{\frac{2q}{q-2}} \leqslant C\|\omega\|_{\frac{2q}{q-2}}.$$

Applying the Sobolev inequality, for $q > 2$ and $q\delta > 2$

$$\|f\|_{\frac{2q}{q-2}} \leqslant C\|f\|_2^{1-\frac{2}{q\delta}} \left\|\Lambda^\delta f\right\|_2^{\frac{2}{q\delta}},$$

and Young's inequality, we obtain

$$\begin{aligned} |J_1| \leqslant{}& \frac{1}{8}\|\partial_x \omega\|_2^2 + \frac{\epsilon}{4}\left\|\Lambda^\delta j\right\|_2^2 + C\|b_1\|_q^{\frac{2q\delta}{q\delta-2}} \|j\|_2^2 \\ &+ \frac{1}{8}\|\partial_x j\|_2^2 + \frac{\epsilon}{4}\left\|\Lambda^\delta \omega\right\|_2^2 + C\|b_1\|_q^{\frac{2q\delta}{q\delta-2}} \|\omega\|_2^2. \end{aligned}$$

J_2 can be bounded through Lemma 1.1,

$$\begin{aligned} |J_2| &\leqslant C\|\partial_x b_1\|_2 \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_{xy} u_2\|_2^{\frac{1}{2}} \|j\|_2^{\frac{1}{2}} \|\partial_x j\|_2^{\frac{1}{2}} \\ &\leqslant \frac{1}{8}\|\partial_x \omega\|_2^2 + \frac{1}{8}\|\partial_x j\|_2^2 + C\|\partial_x b_1\|_2^2 \left(\|\omega\|_2^2 + \|j\|_2^2\right). \end{aligned}$$

To bound J_3, we first integrate by parts to obtain

$$J_3 = -2\int u_1 \partial_{xy} b_1 j - 2\int u_1 \partial_y b_1 \partial_x j.$$

The terms on the right can then be estimated in a similarly fashion as in (6.3) and the bound is

$$|J_3| \leqslant \frac{1}{8}\|\partial_x \omega\|_2^2 + \frac{1}{8}\|\partial_x j\|_2^2 + \frac{\epsilon}{4}\left\|\Lambda^\delta j\right\|_2^2 + C\|u_1\|_q^{\frac{2q\delta}{q\delta-2}} \|j\|_2^2.$$

J_4 can be bounded in a similar fashion as J_2 and

$$|J_4| \leqslant \frac{1}{8}\|\partial_x j\|_2^2 + C\|\partial_x u_1\|_2^2 \|j\|_2^2.$$

Inserting the estimates for J_1, J_2, J_3 and J_4 in (6.2) yields the desired global H^1-bound.

To establish the global H^2-bound, we take the inner product of (6.1) with $(\Delta\omega, \Delta j)$ to obtain, after integration by parts,

$$\frac{1}{2}\frac{d}{dt}\big(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2\big)+\|\nabla\partial_x\omega\|_2^2+\|\nabla\partial_x j\|_2^2+\epsilon\big\|\Lambda^{\delta+1}\omega\big\|_2^2+\epsilon\big\|\Lambda^{\delta+1}j\big\|_2^2$$
$$=L_1+L_2+L_3+L_4+L_5, \tag{6.4}$$

where

$$L_1=-\int\nabla\omega\cdot\nabla u\cdot\nabla\omega\,dx\,dy,$$
$$L_2=-\int\nabla j\cdot\nabla u\cdot\nabla j\,dx\,dy,$$
$$L_3=\int\nabla\omega\cdot\big(\nabla b+(\nabla b)^t\big)\cdot\nabla j\,dx\,dy,$$
$$L_4=2\int\nabla\big[\partial_x b_1(\partial_x u_2+\partial_y u_1)\big]\cdot\nabla j\,dx\,dy,$$
$$L_5=-2\int\nabla\big[\partial_x u_1(\partial_x b_2+\partial_y b_1)\big]\cdot\nabla j\,dx\,dy.$$

To estimate L_1, we write the integrand explicitly

$$L_1=\int\big(\partial_x u_1(\partial_x\omega)^2+(\partial_x u_2+\partial_y u_1)\partial_x\omega\partial_y\omega+\partial_y u_2(\partial_y\omega)^2\big)\,dx\,dy.$$

Each one of them can be bounded by Lemma 1.1 and then by Young's inequality. For example,

$$\int\partial_x u_1(\partial_x\omega)^2\,dx\,dy\leqslant C\|\partial_x u_1\|_2\|\partial_x\omega\|_2^{\frac{1}{2}}\big\|\partial_x^2\omega\big\|_2^{\frac{1}{2}}\|\partial_x\omega\|_2^{\frac{1}{2}}\big\|\partial_{xy}^2\omega\big\|_2^{\frac{1}{2}}$$
$$\leqslant\frac{1}{32}\|\nabla\partial_x\omega\|_2^2+C\|\omega\|_2^2\|\nabla\omega\|_2^2.$$

Since the estimates for other terms are similar, we obtain

$$|L_1|\leqslant\frac{1}{8}\|\nabla\partial_x\omega\|_2^2+C\|\omega\|_2^2\|\nabla\omega\|_2^2+C\|\nabla\omega\|_2^2\|\omega\|_2^{\frac{2}{3}}\|\partial_x\omega\|_2^{\frac{2}{3}}.$$

Similarly, L_2, L_3, L_4 and L_5 are bounded by

$$|L_2|\leqslant\frac{1}{8}\|\nabla\partial_x j\|_2^2+C\big(\|\omega\|_2^2+\|\omega\|_2^{\frac{2}{3}}\|\partial_x\omega\|_2^{\frac{2}{3}}\big)\|\nabla j\|_2^2,$$
$$|L_3|\leqslant\frac{1}{8}\|\nabla\partial_x\omega\|_2^2+\frac{1}{8}\|\nabla\partial_x j\|_2^2$$
$$+C\|j\|_2^2\big(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2\big)+C\|\partial_x j\|_2^2\|\nabla j\|_2^2,$$
$$|L_4|\leqslant\frac{1}{8}\|\nabla\partial_x\omega\|_2^2+\frac{1}{8}\|\nabla\partial_x j\|_2^2$$
$$+C\big(\|j\|_2^2+\|\omega\|_2^2+\|\partial_x j\|_2^2+\|\omega\|_2^{\frac{2}{3}}\|\partial_x\omega\|_2^{\frac{2}{3}}\big)\big(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2\big),$$
$$|L_5|\leqslant\frac{1}{8}\|\nabla\partial_x\omega\|_2^2+\frac{1}{8}\|\nabla\partial_x j\|_2^2$$
$$+C\big(\|j\|_2^2+\|\omega\|_2^2+\|\partial_x j\|_2^2+\|\partial_x\omega\|_2^2\big)\big(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2\big).$$

Inserting these estimates in (6.4), applying Gronwall's inequality and invoking the global H^1-bound, we obtained the desired global H^2-bound for the solution. This concludes the proof of Theorem 6.1. □

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