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Stability and exponential decay for magnetohydrodynamic equations

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This paper focuses on a 2D magnetohydrodynamic system with only horizontal dissipation in the domain $\Omega = \mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being a periodic box. The goal here is to understand the stability problem on perturbations near the background magnetic field (1, 0). Due to the lack of vertical dissipation, this stability problem is difficult. This paper solves the desired stability problem by simultaneously exploiting two smoothing and stabilizing mechanisms: the enhanced dissipation due to the coupling between the velocity and the magnetic fields, and the strong Poincaré type inequalities for the oscillation part of the solution, namely the difference between the solution and its horizontal average. In addition, the oscillation part of the solution is shown to converge exponentially to zero in H^1 as $t \to \infty$. As a consequence, the solution converges to its horizontal average asymptotically.

Keywords: Magnetohydrodynamic equations; partial dissipation; stability; wave equations

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1. Introduction

Let $\Omega = \mathbb{T} \times \mathbb{R}$ with $\mathbb{T} = [0, 1]$ being a one-dimensional (1D) periodic domain and $\mathbb{R}$ being the real line. Consider the 2D incompressible magnetohydrodynamic (MHD) equations with horizontal dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u = -\nabla P + \nu\, \partial_{11} u + B \cdot \nabla B, \quad x \in \Omega, \quad t > 0, \\ \partial_t B + u \cdot \nabla B + \eta B = B \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot B = 0, \\ u(x,0) = u_0(x), \quad B(x,0) = B_0(x), \end{cases} \tag{1.1}$$

where u denotes the velocity field, B the magnetic field and P the pressure, and $\nu > 0$ and η are the viscosity and the damping coefficients, respectively. Here the velocity u obeys a degenerate Navier–Stokes equation with only horizontal dissipation $\nu\partial_{11}u$ and with a Lorentz forcing term. The magnetic field B satisfies the induction equation with a damping term. The goal of this paper is to understand the stability and the large-time behaviour of perturbations near a background magnetic field.

This study is partially motivated by the stabilizing phenomenon of a background magnetic field on electrically conducting fluids that has been observed in physical experiments and numerical simulations (see, e.g., [**1**, **2**, **6**, **12**–**14**, **24**, **25**]). Since the dynamics of electrically conducting fluids is governed by the MHD equations (see, e.g., [**4**, **39**]), the aim here is to establish this remarkable observation as a mathematically rigorous fact on the MHD equations.

We take the background magnetic field to be the unit vector in the x_1-direction, $B^{(0)} = (1,\, 0)$. The corresponding steady-state solution of (1.1) is given by

$$u^{(0)} = (0,0), \quad B^{(0)} = (1,0).$$

We write $(u,\, b)$ with $b = B - B^{(0)}$ for the perturbation near $(u^{(0)},\, B^{(0)})$. Our attention will be focussed on the following new system

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{11} u + b \cdot \nabla b + \partial_1 b, \quad x \in \Omega, \quad t > 0 \\ \partial_t b + u \cdot \nabla b + \eta b = b \cdot \nabla u + \partial_1 u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x),\ b(x,0) = b_0(x). \end{cases} \tag{1.2}$$

In comparison with the original system in (1.1), there are two extra terms $\partial_1 b$ and $\partial_1 u$ in (1.2). We aim to achieve a complete understanding on the stability of solutions to (1.2) in the Sobolev setting. In addition, we also attempt to obtain the precise large-time behaviour of $(u,\, b)$ and establish the eventual dynamics of (1.2).

Due to the lack of vertical dissipation in (1.2), the resolution of the stability problem is not direct. If we follow the standard energy method approach, the difficulty is immediate. The divergence-free conditions $\nabla \cdot u = \nabla \cdot b = 0$ allow us to obtain a suitable upper bound on the H^1-norm of $(u,\, b)$, but it does not appear to be possible to control the H^2-norm directly. Even if we completely ignore the terms related to the magnetic field and simply consider the 2D anisotropic Navier–Stokes

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equations

$$\partial_t u + u \cdot \nabla u + \nabla P = \nu \partial_{11} u,$$

direct energy estimates fail to generate a suitable H^2-bound. In fact, when we resort to the corresponding vorticity formulation

$$\partial_t \omega + u \cdot \nabla \omega = \nu \partial_{11} \omega,$$

the one-directional dissipation is insufficient to bound the nonlinearity directly. In the estimate of $\nabla\omega$,

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_{L^2}^2 + \nu\|\partial_1\nabla\omega\|_{L^2}^2 = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx,$$

the right-hand side does not admit a suitable upper bound. In fact,

$$\begin{aligned} \int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dx &= \int \partial_1 u_1 (\partial_1\omega)^2 \, dx + \int \partial_1 u_2 \partial_1\omega \partial_2\omega \, dx \\ &\quad + \int \partial_2 u_1 \partial_1\omega \, \partial_2\omega \, dx + \int \partial_2 u_2 (\partial_2\omega)^2 \, dx \end{aligned} \tag{1.3}$$

and the two terms in (1.3) can not be controlled suitably.

One novel idea to overcome this difficulty is to explore the stabilizing effect of the magnetic field on the fluids as hinted by the aforementioned experimental results. Mathematically we make full use of the coupling and interaction in the MHD system in (1.2) to unearth the hidden smoothing and stabilizing properties. To do so, we first apply the Leray projection $\mathbb{P} = I - \nabla\Delta^{-1}\nabla\cdot$ to the velocity equation to eliminate the pressure,

$$\partial_t u = \nu\partial_{11} u + \partial_1 b + \mathbb{P}(b \cdot \nabla b - u \cdot \nabla u).$$

By differentiating the linearized system in time

$$\begin{cases} \partial_t u = \nu\partial_{11} u + \partial_1 b, \\ \partial_t b = -\eta b + \partial_1 u \end{cases} \tag{1.4}$$

and making several substitutions, we can convert (1.4) into a system of wave equations

$$\begin{cases} \partial_{tt} u + (\eta + \nu\partial_{11})\partial_t u - (1 + \nu\eta)\partial_{11} u = 0, \\ \partial_{tt} b + (\eta + \nu\partial_{11})\partial_t b - (1 + \nu\eta)\partial_{11} b = 0. \end{cases} \tag{1.5}$$

(1.5) allows us to decouple u and b and exhibits more smoothing and stabilizing properties than (1.4). In particular, both u and b gain weak horizontal dissipation as can be seen from the pieces $(1 + \nu\eta)\partial_{11} u$ and $(1 + \nu\eta)\partial_{11} b$. Unfortunately, this extra regularization does not appear to help with the deficiency of vertical dissipation in the velocity equation. As a consequence, this approach fails.

We remark that a previous work of Feng, Hafeez and Wu [**23**] explored the extra stabilizing and smoothing of the wave structure, and successfully resolved the stability problem on the same MHD system near the background magnetic field

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$B^{(0)} = (0, 1)$. When the background magnetic field is $(0, 1)$, the extra regularity is in the vertical direction and complements with the horizontal dissipation in the velocity equation. Therefore, the direction of the background magnetic field plays a crucial role in the stabilizing phenomenon on electrically conduction fluids.

This paper seeks a different approach to resolve the stability problem concerned here. The spatial domain here is $\Omega = \mathbb{T} \times \mathbb{R}$ and we take full advantage of the geometry of this domain. The horizontal direction is periodic and we can separate the zeroth Fourier mode from the non-zero ones. The zeroth Fourier mode corresponds to the horizontal average. This hints the decomposition of the physical quantities into the horizontal averages and the corresponding oscillation parts. More precisely, for a function f that is integrable in $x \in \mathbb{T}$, we define

$$\bar{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1, \quad f = \bar{f} + \widetilde{f}.$$

This decomposition is orthogonal in the Sobolev space $H^k(\Omega)$ for any integer $k \geqslant 0$ (see lemma 2.2 in § 2). More crucially, the oscillation part $\widetilde{f}$ obeys a strong version of the Poincaré type inequality

$$\|\widetilde{f}\|_{L^2(\Omega)} \leqslant C\, \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}.$$

This inequality allows us to control some of the nonlinear parts in terms of the horizontal dissipation. By invoking the decompositions

$$u = \bar{u} + \widetilde{u}, \quad b = \bar{b} + \widetilde{b}$$

and applying the aforementioned Poincaré inequality together with various anisotropic inequalities, we are able to successfully bound the nonlinearity and establish the following stability result.

Theorem 1.1. *Let $\eta > 0$ and $\nu > 0$. Consider (1.2) with the initial data $(u_0, b_0) \in H^3(\Omega)$, and $\nabla \cdot u_0 = \nabla \cdot b_0 = 0$. Then there exists a constant $\varepsilon = \varepsilon(\nu, \eta) > 0$ such that, if*

$$\|u_0\|_{H^3} + \|b_0\|_{H^3} \leqslant \varepsilon,$$

then (1.2) has a unique global classical solution (u, b) satisfying, for any $t > 0$,

$$\|u(t)\|_{H^3}^2 + \|b(t)\|_{H^3}^2 + \int_0^t \left(\|\partial_1 u\|_{H^3}^2 + \|b\|_{H^3}^2 \right)\, d\tau \leqslant C\, \varepsilon^2,$$

where $C > 0$ is independent of ε and t.

Theorem 1.1 successfully resolves the stability problem on a partially dissipated MHD system near a background magnetic field even when the smoothing effect of the magnetic field is not sufficient to deal with the dissipation deficiency.

Efforts are also devoted to understanding the precise large-time behaviour of the perturbation. We expect the horizontal average $(\bar{u}, \bar{b})$ to behave differently from the oscillation part $(\widetilde{u}, \widetilde{b})$. Intuitively $(\bar{u}, \bar{b})$ corresponds to the zeroth horizontal Fourier mode and the associated dissipation term vanishes. Thus $(\bar{u}, \bar{b})$ may not

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decay in time. In contrast, $(\widetilde{u}, \widetilde{b})$ consists of non-zero horizontal Fourier modes and the horizontal dissipation effectively plays the role of damping. As a consequence, $(\widetilde{u}, \widetilde{b})$ could decay exponentially in time. Our second theorem rigorously confirms this intuition.

THEOREM 1.2. *Let u_0, $b_0 \in H^3(\Omega)$ with $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Assume that $\|u_0\|_{H^3} + \|b_0\|_{H^3} \leqslant \varepsilon$ for sufficiently small $\varepsilon > 0$. Let (u, b) be the corresponding solution of (1.2). Then the H^1 norm of the oscillation part $(\widetilde{u}, \widetilde{b})$ decays exponentially in time,*

$$\|\widetilde{u}(t)\|_{H^1} + \|\widetilde{b}(t)\|_{H^1} \leqslant (\|u_0\|_{H^1} + \|b_0\|_{H^1})e^{-C_1 t}, \tag{1.6}$$

for some constant $C_1 > 0$ and for all $t > 0$.

We explain the main lines in the proof of theorem 1.1. The local well-posedness of (1.2) in the Sobolev space $H^3(\Omega)$ can be shown via standard procedures such as the approach in the book of Majda and Bertozzi [**37**]. Our attention is focussed on the global bound of (u, b) in $H^3(\Omega)$. One of the most suitable tools for this purpose is the bootstrapping argument [**45**]. To set up the argument, we first construct the energy functional. For the MHD system in (1.2), the energy functional $E(t)$ is naturally given by the H^3-norm of (u, b) together with the time integrals from dissipative and damping terms, namely

$$E(t) = \sup_{0\leqslant\tau\leqslant t} \{\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2\} + 2\nu \int_0^t \|\partial_1 u\|_{H^3}^2 \, d\tau + 2\eta \int_0^t \|b\|_{H^3}^2 \, d\tau.$$

The main effort is then devoted to proving the energy inequality

$$E(t) \leqslant E(0) + \ CE^{\frac{3}{2}}(t). \tag{1.7}$$

Once (1.7) is at our disposal, the bootstrapping argument then implies that, if $E(0) := \|(u_0, b_0)\|_{H^3}^2$ is sufficiently small, say

$$\|(u_0, b_0)\|_{H^3} \leqslant \varepsilon$$

for some suitable $\varepsilon > 0$, then $E(t)$ remains uniformly bounded for any $t > 0$,

$$E(t) \leqslant C\,\varepsilon^2,$$

which gives us the desired global bound on $\|(u(t), b(t))\|_{H^3}$. To prove (1.7), we invoke the orthogonal decompositions $u = \bar{u} + \widetilde{u}$ and $b = \bar{b} + \widetilde{b}$, apply the Poincaré type inequalities and anisotropic upper bounds for triple products. More technical details are provided in § 3.

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To prove theorem 1.2, we first take the horizontal average of (1.2) to obtain the equations of $(\bar{u}, \bar{b})$,

$$\begin{cases} \partial_t \bar{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \bar{p} \end{pmatrix} = \overline{b \cdot \nabla \widetilde{b}}, \\ \partial_t \bar{b} + \overline{u \cdot \nabla \widetilde{b}} + \eta \bar{b} = \overline{b \cdot \nabla \widetilde{u}}. \end{cases} \tag{1.8}$$

We then write the equations of $(\widetilde{u}, \widetilde{b})$ by taking the difference of (1.2) and (1.8),

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + u_2 \partial_2 \bar{u} + \nabla \widetilde{p} - \nu \partial_1^2 \widetilde{u} - \widetilde{b \cdot \nabla \widetilde{b}} - b_2 \partial_2 \bar{b} - \partial_1 \widetilde{b} = 0, \\ \partial_t \widetilde{b} + \widetilde{u \cdot \nabla \widetilde{b}} + u_2 \partial_2 \bar{b} + \eta \widetilde{b} - \widetilde{b \cdot \nabla \widetilde{u}} - b_2 \partial_2 \bar{u} - \partial_1 \widetilde{u} = 0. \end{cases} \tag{1.9}$$

The proof of (1.6) is divided into the estimates of $\|(\widetilde{u}, \widetilde{b})\|_{L^2}$ and $\|(\nabla \widetilde{u}, \nabla \widetilde{b})\|_{L^2}$. The efforts are devoted to bounding the nonlinearity in terms of the horizontal derivatives of $\widetilde{u}$. Poincaré's inequality and anisotropic upper bounds for the triple products are used extensively. After a tedious process of evaluating many terms, we obtain

$$\begin{aligned} &\frac{d}{dt}(\|\widetilde{u}\|_{H^1}^2 + \|\widetilde{b}\|_{H^1}^2) + (2\nu - C\|(u,b)\|_{H^3})\|\partial_1 \widetilde{u}\|_{H^1}^2 \\ &\quad + (2\eta - C\|(u,b)\|_{H^3})\|\widetilde{b}\|_{H^1} \leqslant 0, \end{aligned}$$

which yields the decay rate in (1.6). A detailed proof is provided in § 4.

Finally we briefly summarize some of related results to provide a broader view on the studies of the MHD equations. Fundamental issues on the MHD equations such as well-posedness and stability problems have attracted a lot of attention. Substantial progress has recently been made on the well-posedness problem concerning the MHD equations with various partial or fractional dissipation (see, e.g., [**8**–**10**, **16**, **17**, **19**–**22**, **30**, **32**, **35**, **42**, **43**, **46**, **49**, **53**, **55**–**61**]). Since the pioneering work of Alfvén [**2**], the stability problem on various MHD systems has recently gained renewed interests and there are substantial developments. By taking advantage of the Elsässer variables, several papers have successfully solved the stability problem on the ideal MHD equations or the fully dissipated MHD equations with identical (or almost identical) viscosity and magnetic diffusivity (see [**3**, **7**, **26**, **47**]). The stability problem on the MHD equations with only kinematic dissipation in $\mathbb{R}^2$ or $\mathbb{R}^3$ have been solved via different approaches [**15**, **27**, **28**, **33**, **34**, **40**, **41**, **44**, **50**, **51**, **62**, **63**]. The same problem in the periodic setting $\mathbb{T}^3$ has been investigated by [**38**]. The MHD equations with only magnetic diffusivity have recently been studied for the small data global well-posedness near the trivial solution or a background magnetic field [**11**, **29**, **48**, **54**, **64**], although a complete solution on the stability problem near a background magnetic field is currently lacking. When the velocity equation involves only horizontal or vertical dissipation, the velocity equation itself alone may not be stable and the stability problem relies on the enhanced dissipation resulting from the coupling and interaction. Several such MHD systems with degenerate velocity dissipation have been shown to be stable near suitable background magnetic fields [**5**, **23**, **31**, **36**, **52**].

The rest of this paper is divided into three sections. Section 2 states several properties on the aforementioned decomposition and provides several anisotropic inequalities. Section 3 proves theorem 1.1 while § 4 presents the proof of theorem 1.2.

2. Preliminaries

This section states several properties on the decomposition defined in the introduction and provides several anisotropic inequalities to be used in the proofs of theorems 1.1 and 1.2. Some of the materials presented here can be found in [**9**, **18**].

We start by recalling the definition of the horizontal average and the oscillation part. Let $\Omega = \mathbb{T} \times \mathbb{R}$ and let $f = f(x_1, x_2)$ with $(x_1, x_2) \in \Omega$ be sufficiently smooth, say integrable in $x_1 \in \mathbb{T}$. The horizontal average $\overline{f}$ is given by

$$\overline{f}(x_2) = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1. \tag{2.1}$$

We decompose f into $\overline{f}$ and the oscillation portion $\widetilde{f}$,

$$f = \overline{f} + \widetilde{f}. \tag{2.2}$$

The following lemma is a direct consequence of (2.1) and (2.2).

Lemma 2.1. *The average operator and the oscillation operator commute with the partial derivatives, for $i = 1, 2$,*

$$\partial_i \overline{f} = \overline{\partial_i f}, \quad \partial_i \widetilde{f} = \widetilde{\partial_i f}, \quad \partial_1 \overline{f} = 0, \quad \overline{\widetilde{f}} = 0,$$

As a special consequence, if $\nabla \cdot f = 0$, then

$$\nabla \cdot \overline{f} = 0, \quad \nabla \cdot \widetilde{f} = 0.$$

The second lemma states that the decomposition in (2.2) is orthogonal in any Sobolev space $\dot{H}^k(\Omega)$.

Lemma 2.2. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. Let $k \geqslant 0$ be an integer. Let $f \in \dot{H}^k(\Omega)$. Then $\overline{f}$ and $\widetilde{f}$ are orthogonal in $\dot{H}^k(\Omega)$, namely*

$$(\overline{f}, \widetilde{f})_{\dot{H}^k} := \int_\Omega D^k \overline{f} \cdot D^k \widetilde{f}\, dx = 0. \qquad \|f\|^2_{\dot{H}^k(\Omega)} = \|\overline{f}\|^2_{\dot{H}^k(\Omega)} + \|\widetilde{f}\|^2_{\dot{H}^k(\Omega)}$$

In particular, $\|\overline{f}\|_{\dot{H}^k} \leqslant \|f\|_{\dot{H}^k}$ and $\|\widetilde{f}\|_{\dot{H}^k} \leqslant \|f\|_{\dot{H}^k}$.

The oscillation part obeys the following Poincaré type inequalities.

Lemma 2.3. *If $\|\partial_1 \widetilde{f}\|_{L^2(\Omega)} < \infty$, then*

$$\|\widetilde{f}\|_{L^2(\Omega)} \leqslant C \|\partial_1 \widetilde{f}\|_{L^2(\Omega)}.$$

In addition, if $\|\partial_1 \widetilde{f}\|_{H^1(\Omega)} < \infty$, then

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leqslant C \|\partial_1 \widetilde{f}\|_{H^1(\Omega)}.$$

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Proof of lemma 2.3. Since the horizontal average of $\widetilde{f}$ is zero, for any fixed $x_2 \in \mathbb{R}$, there is $a \in \mathbb{T}$ such that

$$\widetilde{f}(a, x_2) = 0.$$

Then, for any $(x_1,\, x_2) \in \Omega$,

$$\widetilde{f}(x_1, x_2) = \int_a^{x_1} \partial_z \widetilde{f}(z, x_2)\, dz \leqslant \int_{\mathbb{T}} |\partial_z \widetilde{f}(z, x_2)|\, dz \leqslant \|\partial_{x_1} \widetilde{f}\|_{L^2(\mathbb{T})}. \tag{2.3}$$

Squaring each side of (2.3) and integrating over Ω yields the first inequality. The second inequality is obtained by taking the $L^\infty(\Omega)$ in (2.3) and using the simple fact that $\|f\|_{L^\infty(\mathbb{R})} \leqslant C\, \|f\|_{H^1(\mathbb{R})}$ for any 1D function $f \in H^1(\mathbb{R})$. □

Next we present several anisotropic inequalities. Anisotropic upper bounds for triple products are frequently used to bound the nonlinear terms when only partial dissipation is present. In the case when the spatial domain is the whole space $\mathbb{R}^2$, Cao and Wu [9] showed and applied the following inequality

$$\left| \int_{\mathbb{R}^2} f\, g\, h \right| \leqslant C\, \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\, \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\, \|g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\, \|\partial_2 g\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\, \|h\|_{L^2(\mathbb{R}^2)}. \tag{2.4}$$

(2.4) is a consequence of the elementary 1D inequality

$$\|f\|_{L^\infty(\mathbb{R})} \leqslant \sqrt{2} \|f\|_{L^2(\mathbb{R})}^{\frac{1}{2}}\, \|f'\|_{L^2(\mathbb{R})}^{\frac{1}{2}}. \tag{2.5}$$

Another consequence of (2.5) is the following inequality

$$\|f\|_{L^\infty(\mathbb{R}^2)} \leqslant C \|f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\, \|\partial_1 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}\, \|\partial_1 \partial_2 f\|_{L^2(\mathbb{R}^2)}^{\frac{1}{4}}.$$

When the 1D spatial domain is a bounded domain, say $\mathbb{T}$,

$$\|f\|_{L^\infty(\mathbb{T})} \leqslant C\, \|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \left(\|f\|_{L^2(\mathbb{T})} + \|f'\|_{L^2(\mathbb{T})} \right)^{\frac{1}{2}}.$$

Since the oscillation part $\widetilde{f}$ has mean zero, for $\widetilde{f} \in H^1(\mathbb{T})$,

$$\|\widetilde{f}\|_{L^\infty(\mathbb{T})} \leqslant C \|\widetilde{f}\|_{L^2(\mathbb{T})}^{\frac{1}{2}} \|(\widetilde{f})'\|_{L^2(\mathbb{T})}^{\frac{1}{2}}.$$

As a consequence of these elementary inequalities, the following two lemmas hold.

LEMMA 2.4. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any $f,\, g,\, h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\int_\Omega |fgh|\; dx \leqslant C \|f\|_{L^2}^{\frac{1}{2}} (\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}} \|g\|_{L^2}^{\frac{1}{2}} \|\partial_2 g\|_{L^2}^{\frac{1}{2}} \|h\|_{L^2}.$$

For any $f \in H^2(\Omega)$, we have

$$\|f\|_{L^\infty(\Omega)} \leqslant C \|f\|_{L^2(\Omega)}^{\frac{1}{4}} (\|f\|_{L^2(\Omega)} + \|\partial_1 f\|_{L^2(\Omega)})^{\frac{1}{4}} \|\partial_2 f\|_{L^2(\Omega)}^{\frac{1}{4}} \times (\|\partial_2 f\|_{L^2(\Omega)} + \|\partial_1 \partial_2 f\|_{L^2(\Omega)})^{\frac{1}{4}}.$$

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After replacing f by the oscillation part, we have the following inequalities.

LEMMA 2.5. *Let $\Omega = \mathbb{T} \times \mathbb{R}$. For any f, g, $h \in L^2(\Omega)$ with $\partial_1 f \in L^2(\Omega)$ and $\partial_2 g \in L^2(\Omega)$, then*

$$\int_\Omega |\widetilde{f}gh| \; dx \leqslant C\|\widetilde{f}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{f}\|_{L^2}^{\frac{1}{2}}\|g\|_{L^2}^{\frac{1}{2}}\|\partial_2 g\|_{L^2}^{\frac{1}{2}}\|h\|_{L^2}.$$

For any $f \in H^2(\Omega)$, we have

$$\|\widetilde{f}\|_{L^\infty(\Omega)} \leqslant C\|\widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{4}}\|\partial_1\widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{4}}\|\partial_2\widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{4}}\|\partial_1\partial_2\widetilde{f}\|_{L^2(\Omega)}^{\frac{1}{4}}.$$

3. Stability

This section is devoted to the proof of theorem 1.1 on the stability of (1.2).

Proof of theorem 1.1. The local well-posedness of (1.2) in the Sobolev space $H^3(\Omega)$ can be shown via standard procedures such as the approach in the book of Majda and Bertozzi [**37**]. Our attention is focussed on the global bound of (u, b) in $H^3(\Omega)$.

The framework of the proof is the bootstrapping argument. To proceed, we define the energy functional as

$$E(t) = \sup_{0\leqslant\tau\leqslant t}\{\|u(\tau)\|_{H^3}^2 + \|b(\tau)\|_{H^3}^2\} + 2\nu\int_0^t \|\partial_1 u\|_{H^3}^2 \; d\tau + 2\eta\int_0^t \|b\|_{H^3}^2 \; d\tau. \quad (3.1)$$

Our main efforts are devoted to proving the following energy inequality

$$E(t) \leqslant E(0) + \; CE^{\frac{3}{2}}(t). \quad (3.2)$$

As we explain later, a direct application of the bootstrapping argument to (3.2) implies the desired global uniform bound on $\|(u, b)\|_{H^3}$.

Attention is first focussed on proving (3.1). Due to the equivalence of the inhomogeneous norm $\|(u, b)\|_{H^3}$ with the sum of the L^2-norm and the homogeneous norm $\|(u, b)\|_{\mathring{H}^3}$, it suffices to bound the homogeneous norm $\|(u, b)\|_{\mathring{H}^3}$. The uniform L^2-bound is an easy consequence of the system in (1.2) itself. Taking the inner product of (1.2) with (u, b), we obtain, after integrating by parts and using $\nabla \cdot u = \nabla \cdot b = 0$,

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\nu\int_0^t \|\partial_1 u\|_{L^2}^2 \; d\tau + 2\eta\int_0^t \|b\|_{L^2}^2 \; d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2. \quad (3.3)$$

To estimate the homogeneous norm $\|(u, b)\|_{\mathring{H}^3}$, we apply $\partial_i^3 (i = 1, 2)$ to (1.2) and then dot with $(\partial_i^3 u, \partial_i^3 b)$ to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i=1}^2(\|\partial_i^3 u\|_{L^2}^2 + \|\partial_i^3 b\|_{L^2}^2) + \sum_{i=1}^2 \nu\|\partial_i^3\partial_1 u\|_{L^2}^2 + \sum_{i=1}^2 \eta\|\partial_i^3 b\|_{L^2}^2 \\ &:= J + K + L + M + N,\end{aligned} \quad (3.4)$$

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where

$$
\begin{aligned}
J &= \sum_{i=1}^{2} \int_\Omega \partial_i^3 \partial_1 b \cdot \partial_i^3 u + \partial_i^3 \partial_1 u \cdot \partial_i^3 b \, dx, \\
K &= -\sum_{i=1}^{2} \int_\Omega \partial_i^3 (u \cdot \nabla u) \cdot \partial_i^3 u \, dx, \\
L &= \sum_{i=1}^{2} \int_\Omega (\partial_i^3 (b \cdot \nabla b) - b \cdot \nabla \partial_i^3 b) \cdot \partial_i^3 u \, dx, \\
M &= -\sum_{i=1}^{2} \int_\Omega \partial_i^3 (u \cdot \nabla b) \cdot \partial_i^3 b \, dx, \\
N &= \sum_{i=1}^{2} \int_\Omega (\partial_i^3 (b \cdot \nabla u) - b \cdot \nabla \partial_i^3 u) \cdot \partial_i^3 b \, dx.
\end{aligned}
$$

By integration by parts, $J = 0$. The estimate of K is long and tedious, and is provided in the later part of the proof. To bound L, we decompose it into two parts,

$$
\begin{aligned}
L &= \sum_{i=1}^{2} \left(\int_\Omega \partial_i^3 (b \cdot \nabla b) \cdot \partial_i^3 u \, dx - \int_\Omega b \cdot \nabla \partial_i^3 b \cdot \partial_i^3 u \, dx \right) \\
&= \sum_{i=1}^{2} \sum_{k=1}^{3} C_3^k \int_\Omega \partial_i^k b \cdot \partial_i^{3-k} \nabla b \cdot \partial_i^3 u \, dx \\
&= \sum_{k=1}^{3} C_3^k \int_\Omega \partial_1^k b \cdot \partial_1^{3-k} \nabla b \cdot \partial_1^3 u \, dx + \sum_{k=1}^{3} C_3^k \int_\Omega \partial_2^k b \cdot \partial_2^{3-k} \nabla b \cdot \partial_2^3 u \, dx \\
&= L_1 + L_2,
\end{aligned}
$$

where $C_3^k = \frac{3!}{k!(3-k)!}$ is the binomial coefficient. By lemma 2.1 and lemma 2.5,

$$
\begin{aligned}
L_1 &= \sum_{k=1}^{3} C_3^k \int_\Omega \partial_1^k \widetilde{b} \cdot \partial_1^{3-k} \nabla \widetilde{b} \cdot \partial_1^3 \widetilde{u} \, dx \\
&\lesssim \sum_{k=1}^{2} \|\partial_1^{3-k} \nabla \widetilde{b}\|_{L^2} \|\partial_1^k \widetilde{b}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_1^k \widetilde{b}\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1^3 \widetilde{b}\|_{L^2} \|\nabla \widetilde{b}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \nabla \widetilde{b}\|_{L^2}^{\frac{1}{2}} \|\partial_1^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^3}^2 \|\partial_1 u\|_{H^3}.
\end{aligned}
$$

We further decompose L_2 into three terms,

$$
\begin{aligned}
L_2 &= 3 \int_\Omega \partial_2 b \cdot \partial_2^2 \nabla b \cdot \partial_2^3 u \, dx + 3 \int_\Omega \partial_2^2 b \cdot \partial_2 \nabla b \cdot \partial_2^3 u \, dx + \int_\Omega \partial_2^3 b \cdot \nabla b \cdot \partial_2^3 u \, dx \\
&= L_{2,1} + L_{2,2} + L_{2,3}.
\end{aligned}
$$

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By Hölder's inequality and lemma 2.4,

$$
\begin{aligned}
L_{2,1} &\lesssim \|\partial_2 b\|_{L^\infty}\|\partial_2^2\nabla b\|_{L^2}\|\partial_2^3 u\|_{L^2}\\
&\lesssim (\|\partial_2 b\|_{L^2}^{\frac14}(\|\partial_2 b\|_{L^2}+\|\partial_{12} b\|_{L^2})^{\frac14}\|\partial_2^2 b\|_{L^2}^{\frac14}\\
&\quad\times(\|\partial_2^2 b\|_{L^2}+\|\partial_1\partial_2^2 b\|_{L^2})^{\frac14}\|\partial_2^2\nabla b\|_{L^2}\|\partial_2^3 u\|_{L^2}\\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
L_{2,3} &\lesssim \|\nabla b\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\|\partial_2^3 u\|_{L^2}\\
&\lesssim (\|\nabla b\|_{L^2}^{\frac14}\|\nabla b\|_{L^2}+\|\partial_1\nabla b\|_{L^2})^{\frac14}\|\partial_2\nabla b\|_{L^2}^{\frac14}\\
&\quad\times(\|\partial_2\nabla b\|_{L^2}+\|\partial_1\partial_2\nabla b\|_{L^2})^{\frac14}\|\partial_2^3 b\|_{L^2}\|\partial_2^3 u\|_{L^2}\\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

By lemma 2.1, $\partial_2\overline{b_2}=-\partial_1\overline{b_1}=0$ and lemma 2.5,

$$
\begin{aligned}
L_{2,2} &= 3\int_\Omega \partial_2^2 b\cdot\partial_2\nabla b\cdot\partial_2^3 u\ dx\\
&= 3\left(\int_\Omega \partial_2^2\overline{b_1}\partial_{21}\widetilde{b}\cdot\partial_2^3 u\ dx+\int_\Omega \partial_2^2\widetilde{b_1}\partial_{21}\widetilde{b}\cdot\partial_2^3 u\ dx+\int_\Omega \partial_2^2\widetilde{b_2}\partial_2^2 b\cdot\partial_2^3 u\ dx\right)\\
&\lesssim \|\partial_2^3 u\|_{L^2}\|\partial_{21}\widetilde{b}\|_{L^2}^{\frac12}\|\partial_1\partial_{21}\widetilde{b}\|_{L^2}^{\frac12}\|\partial_2^2\overline{b_1}\|_{L^2}^{\frac12}\|\partial_2\partial_2^2\overline{b_1}\|_{L^2}^{\frac12}\\
&\quad+\|\partial_2^3 u\|_{L^2}\|\partial_{21}\widetilde{b}\|_{L^2}^{\frac12}\|\partial_1\partial_{21}\widetilde{b}\|_{L^2}^{\frac12}\|\partial_2^2\widetilde{b_1}\|_{L^2}^{\frac12}\|\partial_2\partial_2^2\widetilde{b_1}\|_{L^2}^{\frac12}\\
&\quad+\|\partial_2^3 u\|_{L^2}\|\partial_2^2\widetilde{b_2}\|_{L^2}^{\frac12}\|\partial_1\partial_2^2\widetilde{b_2}\|_{L^2}^{\frac12}\|\partial_2^2 b\|_{L^2}^{\frac12}\|\partial_2\partial_2^2 b\|_{L^2}^{\frac12}\\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

Combining the estimates of L_1 and L_2, we obtain

$$
L\lesssim \|\partial_1 u\|_{H^3}\|b\|_{H^3}^2+\|u\|_{H^3}\|b\|_{H^3}^2. \tag{3.5}
$$

Now we estimate M,

$$
\begin{aligned}
M &= -\sum_{i=1}^2\int_\Omega \partial_i^3(u\cdot\nabla b)\cdot\partial_i^3 b\ dx,\\
&= -\int_\Omega \partial_1^3(u\cdot\nabla b)\cdot\partial_1^3 b\ dx-\int_\Omega \partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\ dx,\\
&= M_1+M_2.
\end{aligned}
$$

By lemma 2.1,

$$
\begin{aligned}
M_1 &= -\sum_{k=1}^3 C_3^k\int_\Omega \partial_1^k\widetilde{u}\cdot\partial_1^{3-k}\nabla\widetilde{b}\cdot\partial_1^3\widetilde{b}\ dx-\int_\Omega \widetilde{u}\cdot\partial_1^3\nabla\widetilde{b}\cdot\partial_1^3\widetilde{b}\ dx\\
&= M_{1,1}+M_{1,2}.
\end{aligned}
$$

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By lemma 2.5, Hölder's inequality, and lemma 2.3,

$$\begin{aligned}
M_{1,1} &\lesssim \sum_{k=2}^{3} \|\partial_1^3\widetilde{b}\|_{L^2}\|\partial_1^k\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1^k\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1^{3-k}\nabla\widetilde{b}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1^{3-k}\nabla\widetilde{b}\|_{L^2}^{\frac{1}{2}} \\
&\quad + \|\partial_1\widetilde{u}\|_{L^\infty}\|\partial_1^2\nabla\widetilde{b}\|_{L^2}\|\partial_1^3\widetilde{b}\|_{L^2} \\
&\lesssim \|b\|_{H^3}^2\|\partial_1 u\|_{H^3} + \|\partial_1^2\widetilde{u}\|_{H^1}\|b\|_{H^3}^2 \\
&\lesssim \|b\|_{H^3}^2\|\partial_1 u\|_{H^3}.
\end{aligned}$$

By integration by parts and $\nabla\cdot\widetilde{u}=0$,

$$M_{1,2} = -\int_\Omega \widetilde{u}\cdot\partial_1^3\nabla\widetilde{b}\cdot\partial_1^3\widetilde{b}\,dx = -\frac{1}{2}\int_\Omega \widetilde{u}\cdot\nabla(\partial_1^3\widetilde{b})^2\,dx = 0.$$

To estimate M_2, we split it into four terms,

$$\begin{aligned}
M_2 &= -\int_\Omega \partial_2^3(u\cdot\nabla b)\cdot\partial_2^3 b\,dx, \\
&= -\sum_{k=1}^{3} C_3^k\int_\Omega \partial_2^k u\cdot\partial_2^{3-k}\nabla b\cdot\partial_2^3 b\,dx - \int_\Omega u\cdot\partial_2^3\nabla b\cdot\partial_2^3 b\,dx \\
&= M_{2,1}+M_{2,2}+M_{2,3}+M_{2,4}.
\end{aligned}$$

$M_{2,4}=0$ due to $\nabla\cdot u=0$. By Hölder's inequality and lemma 2.4,

$$\begin{aligned}
M_{2,1} &= -3\int_\Omega \partial_2 u\cdot\partial_2^2\nabla b\cdot\partial_2^3 b\,dx \\
&\lesssim \|\partial_2 u\|_{L^\infty}\|\partial_2^2\nabla b\|_{L^2}\|\partial_2^3 b\|_{L^2} \\
&\lesssim \|\partial_2 u\|_{L^2}^{\frac{1}{4}}(\|\partial_2 u\|_{L^2}+\|\partial_{12}u\|_{L^2})^{\frac{1}{4}}\|\partial_2^2 u\|_{L^2}^{\frac{1}{4}}(\|\partial_2^2 u\|_{L^2}+\|\partial_1\partial_2^2 u\|_{L^2})^{\frac{1}{4}}\|b\|_{H^3}^2 \\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
M_{2,3} &= -\int_\Omega \partial_2^3 u\cdot\nabla b\cdot\partial_2^3 b\,dx \\
&\lesssim \|\nabla b\|_{L^\infty}\|\partial_2^3 u\|_{L^2}\|\partial_2^3 b\|_{L^2} \\
&\lesssim \|\nabla b\|_{L^2}^{\frac{1}{4}}(\|\nabla b\|_{L^2}+\|\partial_1\nabla b\|_{L^2})^{\frac{1}{4}}\|\partial_2\nabla b\|_{L^2}^{\frac{1}{4}}(\|\partial_2\nabla b\|_{L^2} \\
&\quad +\|\partial_1\partial_2\nabla b\|_{L^2})^{\frac{1}{4}}\|\partial_2^3 u\|_{L^2}\|\partial_2^3 b\|_{L^2} \\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}$$

By lemma 2.1, $\partial_2\overline{u_2} = -\partial_1\overline{u_1}$ and lemma 2.5,

$$
\begin{aligned}
M_{2,2} &= -3\int_\Omega \partial_2^2 u\cdot\partial_2\nabla b\cdot\partial_2^3 b\,dx\\
&= -3\left(\int_\Omega \partial_2^2\overline{u_1}\partial_{21}\widetilde{b}\cdot\partial_2^3 b\,dx + \int_\Omega \partial_2^2\widetilde{u_1}\partial_{21}\widetilde{b}\cdot\partial_2^3 b\,dx + \int_\Omega \partial_2^2\widetilde{u_2}\partial_2^2 b\cdot\partial_2^3 b\,dx\right)\\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

Combining the estimates for M_1 and M_2, we obtain

$$
M \lesssim \|\partial_1 u\|_{H^3}\|b\|_{H^3}^2 + \|u\|_{H^3}\|b\|_{H^3}^2. \tag{3.6}
$$

Now we estimate the term N,

$$
\begin{aligned}
N &= \sum_{i=1}^2\left(\int_\Omega \partial_i^3(b\cdot\nabla u)\cdot\partial_i^3 b\,dx - \int_\Omega b\cdot\nabla\partial_i^3 u\cdot\partial_i^3 b\,dx\right)\\
&= \sum_{i=1}^2\sum_{k=1}^3 C_3^k\int_\Omega \partial_i^k b\cdot\partial_i^{3-k}\nabla u\cdot\partial_i^3 b\,dx\\
&= \sum_{k=1}^3 C_3^k\int_\Omega \partial_1^k b\cdot\partial_1^{3-k}\nabla u\cdot\partial_1^3 b\,dx + \sum_{k=1}^3 C_3^k\int_\Omega \partial_2^k b\cdot\partial_2^{3-k}\nabla u\cdot\partial_2^3 b\,dx\\
&= N_1 + N_2.
\end{aligned}
$$

By lemma 2.1, lemma 2.5, Hölder's inequality, and lemma 2.3,

$$
\begin{aligned}
N_1 &= \sum_{k=1}^3 C_3^k\int_\Omega \partial_1^k\widetilde{b}\cdot\partial_1^{3-k}\nabla\widetilde{u}\cdot\partial_1^3\widetilde{b}\,dx\\
&\lesssim \sum_{k=1}^2 \|\partial_1^3\widetilde{b}\|_{L^2}\|\partial_1^{3-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1^{3-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^k\widetilde{b}\|_{L^2}^{\frac12}\|\partial_2\partial_1^k\widetilde{b}\|_{L^2}^{\frac12}\\
&\quad + \|\nabla\widetilde{u}\|_{L^\infty}\|\partial_1^3\widetilde{b}\|_{L_2}\|\partial_1^3\widetilde{b}\|_{L^2}\\
&\lesssim \|b\|_{H^3}^2\|\partial_1 u\|_{H^3} + \|\partial_1\nabla\widetilde{u}\|_{H^1}\|b\|_{H^3}^2 \lesssim \|b\|_{H^3}^2\|\partial_1 u\|_{H^3}.
\end{aligned}
$$

To bound N_2 we further decompose it into three terms as

$$
\begin{aligned}
N_2 &= 3\int_\Omega \partial_2 b\cdot\partial_2^2\nabla u\cdot\partial_2^3 b\,dx + 3\int_\Omega \partial_2^2 b\cdot\partial_2\nabla u\cdot\partial_2^3 b\,dx + \int_\Omega \partial_2^3 b\cdot\nabla u\cdot\partial_2^3 b\,dx\\
&= N_{2,1} + N_{2,2} + N_{2,3}.
\end{aligned}
$$

By Hölder's inequality and lemma 2.4,

$$
\begin{aligned}
N_{2,1} &\lesssim \|\partial_2 b\|_{L^\infty}\|\partial_2^2\nabla u\|_{L^2}\|\partial_2^3 b\|_{L^2}\\
&\lesssim \|\partial_2 b\|_{L^2}^{\frac14}(\|\partial_2 b\|_{L^2} + \|\partial_{12} b\|_{L^2})^{\frac14}\|\partial_2^2 b\|_{L^2}^{\frac14}\\
&\quad\times(\|\partial_2^2 b\|_{L^2} + \|\partial_1\partial_2^2 b\|_{L^2})^{\frac14}\|\partial_2^2\nabla u\|_{L^2}\|\partial_2^3 b\|_{L^2}\\
&\lesssim \|u\|_{H^3}\|b\|_{H^3}^2.
\end{aligned}
$$

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Similarly,

$$\begin{aligned} N_{2,3} \lesssim\ & \|\nabla u\|_{L^\infty}\|\partial_2^3 b\|_{L^2}\|\partial_2^3 b\|_{L^2} \\ \lesssim\ & \|\nabla u\|_{L^2}^{\frac14}(\|\nabla u\|_{L^2}+\|\partial_1\nabla u\|_{L^2})^{\frac14}\|\partial_2\nabla u\|_{L^2}^{\frac14} \\ & \times(\|\partial_2\nabla u\|_{L^2}+\|\partial_1\partial_2\nabla u\|_{L^2})^{\frac14}\|\partial_2^3 b\|_{L^2}^2 \\ \lesssim\ & \|u\|_{H^3}\|b\|_{H^3}^2. \end{aligned}$$

By lemma 2.1, $\partial_2\overline{b_2} = -\partial_1\overline{b_1} = 0$ and lemma 2.5,

$$\begin{aligned} N_{2,2} &= 3\int_\Omega \partial_2^2 b\cdot\partial_2\nabla u\cdot\partial_2^3 b\ dx \\ &= 3\left(\int_\Omega \partial_2^2\overline{b_1}\partial_{21}\widetilde{u}\cdot\partial_2^3 b\ dx + \int_\Omega \partial_2^2\widetilde{b_1}\partial_{21}\widetilde{u}\cdot\partial_2^3 b\ dx + \int_\Omega \partial_2^2\widetilde{b_2}\partial_2^2 u\cdot\partial_2^3 b\ dx\right) \\ &\lesssim \|u\|_{H^3}\|b\|_{H^3}^2. \end{aligned}$$

Combining estimates of N_1 and N_2, we have

$$N \lesssim \|\partial_1 u\|_{H^3}\|b\|_{H^3}^2 + \|u\|_{H^3}\|b\|_{H^3}^2. \tag{3.7}$$

We now turn to the term K. We split K into two terms,

$$\begin{aligned} K &= -\int_\Omega \partial_1^3(u\cdot\nabla u)\cdot\partial_1^3 u\ dx - \int_\Omega \partial_2^3(u\cdot\nabla u)\cdot\partial_2^3 u\ dx, \\ &= K_1 + K_2. \end{aligned}$$

By integration by parts, lemma 2.1, lemma 2.5 and lemma 2.3,

$$\begin{aligned} K_1 &= -\int_\Omega \partial_1^3(u\cdot\nabla u)\cdot\partial_1^3 u\ dx \\ &= \int_\Omega \partial_1^2(u\cdot\nabla u)\cdot\partial_1^4 u\ dx \\ &= \sum_{k=0}^{2} C_2^k\int_\Omega \partial_1^k u\cdot\partial_1^{2-k}\nabla u\cdot\partial_1^4 u\ dx \\ &= \sum_{k=0}^{2} C_2^k\int_\Omega \partial_1^k \widetilde{u}\cdot\partial_1^{2-k}\nabla \widetilde{u}\cdot\partial_1^4 \widetilde{u}\ dx \\ &\lesssim \sum_{k=1}^{2}\|\partial_1^4\widetilde{u}\|_{L^2}\|\partial_1^k\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1^k\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^{2-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1^{2-k}\nabla\widetilde{u}\|_{L^2}^{\frac12} \\ &\quad + \|\partial_1^4\widetilde{u}\|_{L^2}\|\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^2\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1^2\nabla\widetilde{u}\|_{L^2}^{\frac12} \\ &\lesssim \sum_{k=1}^{2}\|\partial_1^4\widetilde{u}\|_{L^2}\|\partial_1^k\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_1^k\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^{2-k}\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1^{2-k}\nabla\widetilde{u}\|_{L^2}^{\frac12} \\ &\quad + \|\partial_1^4\widetilde{u}\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1^2\nabla\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1^2\nabla\widetilde{u}\|_{L^2}^{\frac12}. \end{aligned}$$

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Therefore,

$$K_1 \lesssim \|\partial_1 \widetilde{u}\|_{H^3}^2 \|\widetilde{u}\|_{H^3}. \tag{3.8}$$

To bound K_2, we further decompose it into four terms,

$$\begin{aligned} K_2 &= -\int_\Omega \partial_2^3(u\cdot\nabla u)\cdot\partial_2^3 u\ dx, \\ &= -\sum_{k=0}^{3} C_3^k \int_\Omega \partial_2^k u\cdot\partial_2^{3-k}\nabla u\cdot\partial_2^3 u\ dx \\ &= K_{2,1}+K_{2,2}+K_{2,3}+K_{2,4}. \end{aligned} \tag{3.9}$$

By integration by parts and $\nabla\cdot u = 0$,

$$K_{2,1} = -\int_\Omega u\cdot\partial_2^3\nabla u\cdot\partial_2^3 u\ dx = 0.$$

Next we bound $K_{2,2}$. By lemma 2.1 and $\nabla\cdot u = 0$,

$$\begin{aligned} K_{2,2} &= -3\int_\Omega \partial_2 u\cdot\partial_2^2\nabla u\cdot\partial_2^3 u\ dx \\ &= -3\left(\int_\Omega \partial_2 u_1\partial_2^2\partial_1 u\cdot\partial_2^3 u\ dx + \int_\Omega \partial_2 u_2\partial_2^2\partial_2 u\cdot\partial_2^3 u\ dx\right) \\ &= -3\int_\Omega \partial_2\overline{u_1}\partial_2^2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\ dx - 3\int_\Omega \partial_2\overline{u_1}\partial_2^2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\ dx \\ &\quad - 3\int_\Omega \partial_2\widetilde{u_1}\partial_2^2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\ dx - 3\int_\Omega \partial_2\widetilde{u_1}\partial_2^2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\ dx \\ &\quad + 3\int_\Omega \partial_1\widetilde{u_1}\partial_2^2\partial_2\overline{u}\cdot\partial_2^3\overline{u}\ dx + 3\int_\Omega \partial_1\widetilde{u_1}\partial_2^2\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\ dx \\ &\quad + 3\int_\Omega \partial_1\widetilde{u_1}\partial_2^2\partial_2\widetilde{u}\cdot\partial_2^3\overline{u}\ dx + 3\int_\Omega \partial_1\widetilde{u_1}\partial_2^2\partial_2\widetilde{u}\cdot\partial_2^3\widetilde{u}\ dx \\ &= K_{2,2,1}+K_{2,2,2}+K_{2,2,3}+K_{2,2,4}+K_{2,2,5}+K_{2,2,6}+K_{2,2,7}+K_{2,2,8}. \end{aligned}$$

By integration by parts and lemma 2.1,

$$\begin{aligned} K_{2,2,1} &= -3\int_\Omega \partial_2\overline{u_1}\partial_2^2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\ dx \\ &= 3\left(\int_\Omega \partial_1\partial_2\overline{u_1}\partial_2^2\widetilde{u}\cdot\partial_2^3\overline{u}\ dx + \int_\Omega \partial_2\overline{u_1}\partial_2^2\widetilde{u}\cdot\partial_1\partial_2^3\overline{u}\ dx\right) = 0. \end{aligned}$$

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Similarly, $K_{2,2,5} = 0$. By lemma 2.5 and lemma 2.3,

$$\begin{aligned}K_{2,2,2} &= -3\int_\Omega \partial_2\overline{u_1}\partial_2^2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx\\ &\lesssim \|\partial_2\overline{u_1}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_2\overline{u_1}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.\end{aligned}$$

$K_{2,2,4}$ and $K_{2,2,8}$ can be bounded similarly as $K_{2,2,2}$. By lemma 2.5 and lemma 2.3,

$$\begin{aligned}K_{2,2,3} &= -3\int_\Omega \partial_2\widetilde{u_1}\partial_2^2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx\\ &\lesssim \|\partial_2^3\overline{u}\|_{L^2}\|\partial_2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_2^3\overline{u}\|_{L^2}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.\end{aligned}$$

By lemma 2.5 and lemma 2.3,

$$\begin{aligned}K_{2,2,6} &= 3\int_\Omega \partial_1\widetilde{u_1}\partial_2^2\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\,dx\\ &\lesssim \|\partial_2^2\partial_2\overline{u}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_2^2\partial_2\overline{u}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.\\ K_{2,2,7} &= 3\int_\Omega \partial_1\widetilde{u_1}\partial_2^2\partial_2\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx\\ &\lesssim \|\partial_2^2\partial_2\widetilde{u}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_2^2\partial_2\overline{u}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\\ &\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.\end{aligned}$$

Therefore,

$$K_{2,2} \lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.$$

Now we bound $K_{2,3}$. By lemma 2.1 and the divergence-free condition,

$$\begin{aligned}K_{2,3} &= -3\int_\Omega \partial_2^2 u\cdot\partial_2\nabla u\cdot\partial_2^3 u\,dx\\ &= -3\left(\int_\Omega \partial_2^2 u_1\partial_2\partial_1 u\cdot\partial_2^3 u\,dx + \int_\Omega \partial_2^2 u_2\partial_2\partial_2 u\cdot\partial_2^3 u\,dx\right)\\ &= -3\int_\Omega \partial_2^2\overline{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx - 3\int_\Omega \partial_2^2\overline{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx\end{aligned}$$

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$$
\begin{aligned}
&-3\int_\Omega \partial_2^2\widetilde{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx-3\int_\Omega \partial_2^2\widetilde{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx\\
&+3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\overline{u}\cdot\partial_2^3\overline{u}\,dx+3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\,dx\\
&+3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\widetilde{u}\cdot\partial_2^3\overline{u}\,dx+3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx\\
=&\,K_{2,3,1}+K_{2,3,2}+K_{2,3,3}+K_{2,3,4}+K_{2,3,5}+K_{2,3,6}+K_{2,3,7}+K_{2,3,8}.
\end{aligned}
$$

Clearly $K_{2,3,1}=0$ and $K_{2,3,5}=0$. To bound the remaining terms of $K_{2,3}$ we use lemma 2.5 and lemma 2.3,

$$
\begin{aligned}
K_{2,3,2}&=-3\int_\Omega \partial_2^2\overline{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx\\
&\lesssim \|\partial_2^2\overline{u_1}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_2^2\overline{u_1}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}
$$

$K_{2,3,4}$ and $K_{2,3,7}$ can be bounded similarly,

$$
\begin{aligned}
K_{2,3,3}&=-3\int_\Omega \partial_2^2\widetilde{u_1}\partial_2\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx\\
&\lesssim \|\partial_2^3\overline{u}\|_{L^2}\|\partial_2^2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2^2\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_2^3\overline{u}\|_{L^2}\|\partial_1\partial_2^2\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_1\partial_2^2\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u}\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.\\
K_{2,3,6}&=3\int_\Omega \partial_2\partial_1\widetilde{u_1}\partial_2\partial_2\overline{u}\cdot\partial_2^3\widetilde{u}\,dx\\
&\lesssim \|\partial_2^2\overline{u}\|_{L^2}\|\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_2^2\overline{u}\|_{L^2}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_1\partial_2^3\widetilde{u}\|_{L^2}^{\frac12}\|\partial_2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\|\partial_2^2\partial_1\widetilde{u_1}\|_{L^2}^{\frac12}\\
&\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.
\end{aligned}
$$

$K_{2,3,8}$ can also be bounded similarly. Hence,

$$K_{2,3}\lesssim \|\partial_1 u\|_{H^3}^2\|u\|_{H^3}.$$

Now we bound the last term $K_{2,4}$ in (3.9). By lemma 2.1 and $\nabla\cdot u=0$,

$$
\begin{aligned}
K_{2,4}&=-\int_\Omega \partial_2^3 u\cdot\nabla u\cdot\partial_2^3 u\,dx\\
&=-\left(\int_\Omega \partial_2^3 u_1\partial_1 u\cdot\partial_2^3 u\,dx+\int_\Omega \partial_2^3 u_2\partial_2 u\cdot\partial_2^3 u\,dx\right)\\
&=-\int_\Omega \partial_2^3\overline{u_1}\partial_1\widetilde{u}\cdot\partial_2^3\overline{u}\,dx-\int_\Omega \partial_2^3\overline{u_1}\partial_1\widetilde{u}\cdot\partial_2^3\widetilde{u}\,dx
\end{aligned}
$$

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$$
\begin{aligned}
&- \int_\Omega \partial_2^3 \widetilde{u_1} \partial_1 \widetilde{u} \cdot \partial_2^3 \overline{u}\, dx - \int_\Omega \partial_2^3 \widetilde{u_1} \partial_1 \widetilde{u} \cdot \partial_2^3 \widetilde{u}\, dx \\
&+ \int_\Omega \partial_2^2 \partial_1 \widetilde{u_1} \partial_2 \overline{u} \cdot \partial_2^3 \overline{u}\, dx + \int_\Omega \partial_2^2 \partial_1 \widetilde{u_1} \partial_2 \overline{u} \cdot \partial_2^3 \widetilde{u}\, dx \\
&+ \int_\Omega \partial_2^2 \partial_1 \widetilde{u_1} \partial_2 \widetilde{u} \cdot \partial_2^3 \overline{u}\, dx + \int_\Omega \partial_2^2 \partial_1 \widetilde{u_1} \partial_2 \widetilde{u} \cdot \partial_2^3 \widetilde{u}\, dx \\
= {} & K_{2,4,1} + K_{2,4,2} + K_{2,4,3} + K_{2,4,4} + K_{2,4,5} + K_{2,4,6} + K_{2,4,7} + K_{2,4,8}.
\end{aligned}
$$

Again $K_{2,4,1} = 0$ and $K_{2,4,5} = 0$. To bound the remaining terms of $K_{2,4}$ we use lemma 2.5 and lemma 2.3,

$$
\begin{aligned}
K_{2,4,2} &= - \int_\Omega \partial_2^3 \overline{u_1} \partial_1 \widetilde{u} \cdot \partial_2^3 \widetilde{u}\, dx \\
&\lesssim \|\partial_2^3 \overline{u_1}\|_{L^2} \|\partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^3 \overline{u_1}\|_{L^2} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2 \|u\|_{H^3}.
\end{aligned}
$$

$K_{2,4,4}$ and $K_{2,4,7}$ can be bounded similarly.

$$
\begin{aligned}
K_{2,4,3} &= - \int_\Omega \partial_2^3 \widetilde{u_1} \partial_1 \widetilde{u} \cdot \partial_2^3 \overline{u}\, dx \\
&\lesssim \|\partial_2^3 \overline{u}\|_{L^2} \|\partial_2^3 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2^3 \overline{u}\|_{L^2} \|\partial_1 \partial_2^3 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2 \|u\|_{H^3}. \\
K_{2,4,6} &= \int_\Omega \partial_2^2 \partial_1 \widetilde{u_1} \partial_2 \overline{u} \cdot \partial_2^3 \widetilde{u}\, dx \\
&\lesssim \|\partial_2 \overline{u}\|_{L^2} \|\partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 \overline{u}\|_{L^2} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2^3 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_2^2 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2^3 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_1 u\|_{H^3}^2 \|u\|_{H^3}.
\end{aligned}
$$

$K_{2,4,8}$ can be bounded similarly. Hence,

$$
K_{2,4} \lesssim \|\partial_1 u\|_{H^3}^2 \|u\|_{H^3}.
$$

Putting together the upper bounds for $K_{2,1}$ through $K_{2,4}$, we find

$$
K_2 \lesssim \|\partial_1 u\|_{H^3}^2 \|u\|_{H^3}. \tag{3.10}
$$

Collecting the upper bounds in (3.8) and (3.10) yields

$$
K \lesssim \|\partial_1 u\|_{H^3}^2 \|u\|_{H^3}. \tag{3.11}
$$

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Integrating (3.4) in time and then adding to (3.3), we have, after recalling the definition of E in (3.1),

$$E(t) \leqslant E(0) + \int_0^t (J + K + L + M + N)\, d\tau.$$

Collecting the upper bounds in (3.5), (3.6), (3.7)and (3.11) leads to the desired inequality in (3.2),

$$\begin{aligned} E(t) &\lesssim E(0) + \int_0^t (\|\partial_1 u\|_{H^3}^2 \|u\|_{H^3} + \|\partial_1 u\|_{H^3} \|b\|_{H^3}^2 + \|u\|_{H^3} \|b\|_{H^3}^2)\, d\tau \\ &\leqslant \quad E(0) + C\, E^{\frac{3}{2}}(t). \end{aligned} \tag{3.12}$$

We apply the bootstrapping argument to (3.12). The initial data is taken to be sufficiently small, say

$$\|(u_0, b_0)\|_{H^3} \leqslant \varepsilon$$

with ε satisfying

$$4\,\varepsilon^2 \leqslant \delta_0 := \frac{1}{4C^2}.$$

We make the ansatz that, for $0 \leqslant t \leqslant T$

$$E(t) \leqslant \delta_0.$$

Then (3.12) implies

$$\begin{aligned} E(t) &\leqslant \varepsilon^2 + C E^{\frac{1}{2}}(t)\, E(t) \\ &\leqslant \varepsilon^2 + C\, \frac{1}{2C}\, E(t) \end{aligned}$$

or

$$\frac{1}{2} E(t) \leqslant \varepsilon^2 \quad \text{or} \quad E(t) \leqslant 2\varepsilon^2 = \frac{1}{2}\delta_0.$$

The bootstrapping argument then implies that $T = \infty$ and $E(t) \leqslant \delta_0$. As a consequence, for any $0 \leqslant t \leqslant \infty$,

$$\|(u(t), b(t))\|_{H^3}^2 \leqslant E(t) \leqslant \delta_0.$$

This completes the proof for theorem 1.1. □

4. Proof of theorem 1.2

This section proves theorem 1.2, which assesses that the oscillation part $(\widetilde{u}, \widetilde{b})$ decays exponentially to zero in the H^1-norm as $t \to \infty$. We consider the equations of $(\widetilde{u}, \widetilde{b})$ and apply the properties of the orthogonal decomposition and several anisotropic inequalities.

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Proof of theorem 1.2. We first write the equation of $(\bar{u}, \bar{b})$. By taking the average of (1.2), we have

$$\begin{cases} \partial_t \bar{u} + \overline{u \cdot \nabla \widetilde{u}} + \begin{pmatrix} 0 \\ \partial_2 \bar{p} \end{pmatrix} = \overline{b \cdot \nabla \widetilde{b}}, \\ \partial_t \bar{b} + \overline{u \cdot \nabla \widetilde{b}} + \eta \bar{b} = \overline{b \cdot \nabla \widetilde{u}}. \end{cases} \tag{4.1}$$

Taking the difference of (1.2) and (4.1), we obtain

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u \cdot \nabla \widetilde{u}} + u_2 \partial_2 \bar{u} + \nabla \widetilde{p} - \nu \partial_1^2 \widetilde{u} - \widetilde{b \cdot \nabla \widetilde{b}} - b_2 \partial_2 \bar{b} - \partial_1 \widetilde{b} = 0, \\ \partial_t \widetilde{b} + \widetilde{u \cdot \nabla \widetilde{b}} + u_2 \partial_2 \bar{b} + \eta \widetilde{b} - \widetilde{b \cdot \nabla \widetilde{u}} - b_2 \partial_2 \bar{u} - \partial_1 \widetilde{u} = 0. \end{cases} \tag{4.2}$$

Taking the inner product of (4.2) with $(\widetilde{u}, \widetilde{b})$, after integration by parts and divergence-free conditions, we find

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}\|_{L^2} + \|\widetilde{b}\|_{L^2}) + \nu\|\partial_1 \widetilde{u}\|_{L^2}^2 + \eta\|\widetilde{b}\|_{L^2}^2 \\ &= -\int \widetilde{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u}\, dx - \int u_2 \partial_2 \bar{u} \cdot \widetilde{u}\, dx - \int \widetilde{u \cdot \nabla \widetilde{b}} \cdot \widetilde{b}\, dx \\ &\quad + \int \widetilde{b \cdot \nabla \widetilde{b}} \cdot \widetilde{u}\, dx + \int b_2 \partial_2 \bar{b} \cdot \widetilde{u}\, dx - \int u_2 \partial_2 \bar{b} \cdot \widetilde{b}\, dx \\ &\quad + \int \widetilde{b \cdot \nabla \widetilde{u}} \cdot \widetilde{b}\, dx + \int b_2 \partial_2 \bar{u} \cdot \widetilde{b}\, dx \\ &:= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8. \end{aligned} \tag{4.3}$$

By lemma 2.1,

$$A_1 = -\int u \cdot \nabla \widetilde{u} \cdot \widetilde{u}\, dx + \int \overline{u \cdot \nabla \widetilde{u}} \cdot \widetilde{u}\, dx = 0.$$

Similarly, $A_3 = 0$. By lemma 2.5, lemma 2.3 and the divergence-free conditions,

$$\begin{aligned} A_2 &= -\int \widetilde{u_2} \partial_2 \bar{u} \cdot \widetilde{u}\, dx \\ &\lesssim \|\partial_2 \bar{u}\|_{L^2} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_2 \bar{u}\|_{L^2} \|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|u\|_{H^1} \|\partial_1 \widetilde{u}\|_{L^2}^2. \end{aligned}$$

By lemma 2.1 and the divergence-free conditions,

$$\begin{aligned} A_4 + A_7 &= \int \widetilde{b \cdot \nabla \widetilde{b}} \cdot \widetilde{u}\, dx + \int \widetilde{b \cdot \nabla \widetilde{u}} \cdot \widetilde{b}\, dx \\ &= \int b \cdot \nabla \widetilde{b} \cdot \widetilde{u}\, dx + \int b \cdot \nabla \widetilde{u} \cdot \widetilde{b}\, dx - \int \overline{b \cdot \nabla \widetilde{b}} \cdot \widetilde{u}\, dx \end{aligned}$$

$$
\begin{aligned}
&\quad - \int \overline{b \cdot \nabla \widetilde{u}} \cdot \widetilde{b} \, dx \\
&= 0.
\end{aligned}
$$

By lemma 2.5 and lemma 2.3,

$$
\begin{aligned}
A_5 &= \int b_2 \partial_2 \bar{b} \cdot \widetilde{u} \, dx = \int \widetilde{b_2} \partial_2 \bar{b} \cdot \widetilde{u} \, dx \\
&\lesssim \|\widetilde{b_2}\|_{L^2} \|\partial_2 \bar{b}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \bar{b}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|b\|_{H^2} \|\widetilde{b}\|_{L^2} \|\partial_1 \widetilde{u}\|_{L^2} \\
&\lesssim \|b\|_{H^2} (\|\widetilde{b}\|_{L^2}^2 + \|\partial_1 \widetilde{u}\|_{L^2}^2).
\end{aligned}
$$

Similarly, by lemma 2.5, lemma 2.3 and the divergence-free conditions,

$$
\begin{aligned}
A_6 &= - \int \widetilde{u_2} \partial_2 \bar{b} \cdot \widetilde{b} \, dx \\
&\lesssim \|\widetilde{b}\|_{L^2} \|\partial_2 \bar{b}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_2 \bar{b}\|_{L^2}^{\frac{1}{2}} \|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \widetilde{u_2}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\widetilde{b}\|_{L^2} \|b\|_{H^2} \|\partial_1 \widetilde{u_2}\|_{L^2} \\
&\lesssim \|b\|_{H^2} (\|\widetilde{b}\|_{L^2}^2 + \|\partial_1 \widetilde{u}\|_{L^2}^2).
\end{aligned}
$$

By lemma 2.4 and Hölder's inequality,

$$
\begin{aligned}
A_8 &= \int \widetilde{b_2} \partial_2 \bar{u} \cdot \widetilde{b} \, dx \lesssim \|\partial_2 \bar{u}\|_{L^\infty} \|\widetilde{b}\|_{L^2}^2 \\
&\lesssim \|\partial_2 \bar{u}\|_{L^2}^{\frac{1}{4}} (\|\partial_2 \bar{u}\|_{L^2} + \|\partial_1 \partial_2 \bar{u}\|_{L^2})^{\frac{1}{4}} \|\partial_2 \partial_2 \bar{u}\|_{L^2}^{\frac{1}{4}} \\
&\quad \times (\|\partial_2 \partial_2 \bar{u}\|_{L^2} + \|\partial_1 \partial_2^2 \bar{u}\|_{L^2})^{\frac{1}{4}} \|\widetilde{b}\|_{L^2}^2 \\
&\lesssim \|u\|_{H^3} \|\widetilde{b}\|_{L^2}^2.
\end{aligned} \tag{4.4}
$$

Collecting the estimates for A_1 through A_8 in (4.3), we obtain

$$
\begin{aligned}
&\frac{d}{dt} (\|\widetilde{u}\|_{L^2}^2 + \|\widetilde{b}\|_{L^2}^2) + (2\nu - C\|(u, b)\|_{H^3}) \|\partial_1 \widetilde{u}\|_{L^2}^2 \\
&\quad + (2\eta - C\|(u, b)\|_{H^3}) \|\widetilde{b}\|_{L^2} \leqslant 0.
\end{aligned}
$$

According to theorem 1.1, if $\varepsilon > 0$ is sufficiently small and $\|u_0\|_{H^3} + \|b_0\|_{H^3} \leqslant \varepsilon$, then $\|u\|_{H^3} + \|b\|_{H^3} \leqslant C\varepsilon$ and

$$
2\nu - C\|(u, b)\|_{H^3} \geqslant \nu, \qquad 2\eta - C\|(u, b)\|_{H^3} \geqslant \eta.
$$

By lemma 2.3,

$$
\|\widetilde{u}(t)\|_{L^2} + \|\widetilde{b}(t)\|_{L^2} \leqslant (\|u_0\|_{L^2} + \|b_0\|_{L^2}) e^{-C_1 t}, \tag{4.5}
$$

where $C_1 = C_1(\nu, \eta) > 0$.

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Next we consider the exponential decay for $\|(\nabla\widetilde{u}(t),\, \nabla\widetilde{b}(t))\|_{L^2}$. Taking the gradient of (4.2) yields

$$\begin{cases} \partial_t\nabla\widetilde{u} + \nabla(\widetilde{u\cdot\nabla\widetilde{u}}) + \nabla(u_2\partial_2\bar{u}) + \nabla\nabla\widetilde{p} - \nu\partial_1^2\nabla\widetilde{u} \\ \qquad -\nabla(\widetilde{b\cdot\nabla\widetilde{b}}) - \nabla(b_2\partial_2\bar{b}) - \partial_1\nabla\widetilde{b} = 0, \\ \partial_t\nabla\widetilde{b} + \nabla(\widetilde{u\cdot\nabla\widetilde{b}}) + \nabla(u_2\partial_2\bar{b}) + \eta\nabla\widetilde{b} - \nabla(\widetilde{b\cdot\nabla\widetilde{u}}) \\ \qquad -\nabla(b_2\partial_2\bar{u}) - \partial_1\nabla\widetilde{u} = 0. \end{cases} \tag{4.6}$$

Dotting (4.6) with $(\nabla\widetilde{u},\, \nabla\widetilde{b})$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\nabla\widetilde{u}\|_{L^2} + \|\nabla\widetilde{b}\|_{L^2}) + \nu\|\partial_1\nabla\widetilde{u}\|_{L^2}^2 + \eta\|\nabla\widetilde{b}\|_{L^2}^2 \\ &= -\int \nabla(\widetilde{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}\,dx - \int \nabla(u_2\partial_2\bar{u})\cdot\nabla\widetilde{u}\,dx - \int \nabla(\widetilde{u\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{b}\,dx \\ &\quad + \int \nabla(\widetilde{b\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{u}\,d + \int \nabla(b_2\partial_2\bar{b})\cdot\nabla\widetilde{u}\,dx - \int \nabla(u_2\partial_2\bar{b})\cdot\nabla\widetilde{b}\,dx \\ &\quad + \int \nabla(\widetilde{b\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{b}\,dx + \int \nabla(b_2\partial_2\bar{u})\cdot\nabla\widetilde{b}\,dx \\ &:= B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7 + B_8. \end{aligned} \tag{4.7}$$

By lemma 2.1, B_1 can be written as

$$\begin{aligned} B_1 &= -\int \nabla(u\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{u}\,dx + \int \nabla(\overline{u\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{u}\,dx \\ &= -\int \partial_1 u_1\partial_1\widetilde{u}\cdot\partial_1\widetilde{u}\,dx - \int \partial_1 u_2\partial_2\widetilde{u}\cdot\partial_1\widetilde{u}\,dx \\ &\quad -\int \partial_2 u_1\partial_1\widetilde{u}\cdot\partial_2\widetilde{u}\,dx - \int \partial_2 u_2\partial_2\widetilde{u}\cdot\partial_2\widetilde{u}\,dx \\ &= B_{1,1} + B_{1,2} + B_{1,3} + B_{1,4}. \end{aligned}$$

By lemma 2.5 and lemma 2.3, $B_{1,1}$ can be bounded by

$$\begin{aligned} B_{1,1} &\lesssim \|\partial_1 u_1\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1^2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|\partial_1 u_1\|_{L^2}\|\partial_1^2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1^2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_2\partial_1\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\ &\lesssim \|u\|_{H^3}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2. \end{aligned}$$

$B_{1,2}$ and $B_{1,3}$ can be bounded similarly and

$$B_{1,2},\ B_{1,3} \lesssim \|u\|_{H^3}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.$$

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For $B_{1,4}$, using the divergence-free condition of u and by lemma 2.5, lemma 2.1 and lemma 2.3, we obtain

$$\begin{aligned}
B_{1,4} &= \int \partial_1 u_1 \partial_2 \widetilde{u} \cdot \partial_2 \widetilde{u}\, dx = \int \partial_1 \widetilde{u_1} \partial_2 \widetilde{u} \cdot \partial_2 \widetilde{u}\, dx \\
&\lesssim \|\partial_2 \widetilde{u}\|_{L^2} \|\partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2 \widetilde{u}\|_{L^2} \|\partial_1^2 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_2 \partial_1 \widetilde{u_1}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^3} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2.
\end{aligned}$$

Hence, B_1 is bounded by

$$B_1 \lesssim \|u\|_{H^3} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2.$$

Similarly, we can bound B_3 by lemma 2.4 and Hölder's inequality,

$$\begin{aligned}
B_3 &= -\int \nabla (\widetilde{u \cdot \nabla \widetilde{b}}) \cdot \nabla \widetilde{b}\, dx \\
&= -\int \nabla (u \cdot \nabla \widetilde{b}) \cdot \nabla \widetilde{b}\, dx + \int \nabla (\overline{u \cdot \nabla \widetilde{b}}) \cdot \nabla \widetilde{b}\, dx \\
&= -\int \nabla u \cdot \nabla \widetilde{b} \cdot \nabla \widetilde{b}\, dx \lesssim \|u\|_{H^3} \|\nabla \widetilde{b}\|_{L^2}^2.
\end{aligned}$$

In order to bound B_2, we rewrite it as

$$\begin{aligned}
B_2 &= -\int \nabla (u_2 \partial_2 \bar{u}) \cdot \nabla \widetilde{u}\, dx \\
&= -\int \nabla u_2 \partial_2 \bar{u} \cdot \nabla \widetilde{u}\, dx - \int u_2 \nabla \partial_2 \bar{u} \cdot \nabla \widetilde{u}\, dx \\
&= -\int \partial_1 u_2 \partial_2 \bar{u} \cdot \partial_1 \widetilde{u}\, dx - \int u_2 \partial_1 \partial_2 \bar{u} \cdot \partial_1 \widetilde{u}\, dx \\
&\quad + \int \partial_1 u_1 \partial_2 \bar{u} \cdot \partial_2 \widetilde{u}\, dx - \int u_2 \partial_2 \partial_2 \bar{u} \cdot \partial_2 \widetilde{u}\, dx \\
&:= B_{2,1} + B_{2,2} + B_{2,3} + B_{2,4}.
\end{aligned}$$

According to the definition of $\bar{u}$,

$$B_{2,2} = 0.$$

Using lemma 2.3, Hölder's inequality and proceeding as in (4.4) for $\|\partial_2 \bar{u}\|_{L^\infty}$, we find

$$\begin{aligned}
B_{2,1} &= -\int \partial_1 \widetilde{u_2} \partial_2 \bar{u} \cdot \partial_1 \widetilde{u}\, dx \\
&\lesssim \|\partial_2 \bar{u}\|_{L^\infty} \|\partial_1 \widetilde{u}\|_{L^2}^2 \lesssim \|u\|_{H^3} \|\partial_1 \nabla \widetilde{u}\|_{L^2}^2.
\end{aligned}$$

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Similarly, $B_{2,3}$ has the same bound as $B_{2,1}$. By lemma 2.5 and lemma 2.3,

$$\begin{aligned}
B_{2,4} &= -\int \widetilde{u_2}\partial_2\partial_2\bar{u}\cdot\partial_2\widetilde{u}\,dx \\
&\lesssim \|\partial_2\partial_2\bar{u}\|_{L^2}\|\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2\partial_2\bar{u}\|_{L^2}\|\partial_1\partial_1\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|u\|_{H^3}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.
\end{aligned}$$

Hence, the bound for B_2 is

$$B_2 \lesssim \|u\|_{H^3}\|\partial_1\nabla\widetilde{u}\|_{L^2}^2.$$

Similarly,

$$\begin{aligned}
B_5 &= \int \nabla(b_2\partial_2\bar{b})\cdot\nabla\widetilde{u}\,dx \\
&= -\int \partial_1 b_2\partial_2\bar{b}\cdot\partial_1\widetilde{u}\,dx - \int b_2\partial_1\partial_2\bar{b}\cdot\partial_1\widetilde{u}\,dx \\
&\quad + \int \partial_1 b_1\partial_2\bar{b}\cdot\partial_2\widetilde{u}\,dx - \int b_2\partial_2\partial_2\bar{b}\cdot\partial_2\widetilde{u}\,dx \\
&= B_{5,1} + B_{5,2} + B_{5,3} + B_{5,4}.
\end{aligned}$$

By the definition of $\bar{b}$, $B_{5,2} = 0$. By lemma 2.1, lemma 2.4, lemma 2.3, Hölder's inequality and Young's inequality,

$$\begin{aligned}
B_{5,1} &= -\int \partial_1\widetilde{b_2}\partial_2\bar{b}\cdot\partial_1\widetilde{u}\,dx \\
&\lesssim \|\partial_2\bar{b}\|_{L^\infty}\|\partial_1\widetilde{b_2}\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2} \\
&\lesssim \|\partial_2\bar{b}\|_{L^2}^{\frac{1}{4}}(\|\partial_2\bar{b}\|_{L^2} + \|\partial_1\partial_2\bar{b}\|_{L^2})^{\frac{1}{4}}\|\partial_2\partial_2\bar{b}\|_{L^2}^{\frac{1}{4}} \\
&\quad \times (\|\partial_2\partial_2\bar{b}\|_{L^2} + \|\partial_1\partial_2^2\bar{b}\|_{L^2})^{\frac{1}{4}}\|\partial_1\widetilde{b_2}\|_{L^2}\|\partial_1\partial_1\widetilde{u}\|_{L^2} \\
&\lesssim \|b\|_{H^3}\|\nabla\widetilde{b}\|_{L^2}\|\partial_1\nabla\widetilde{u}\|_{L^2} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).
\end{aligned}$$

Similarly, $B_{5,3}$ obeys the same bound. By lemma 2.5, lemma 2.3 and Young's inequality,

$$\begin{aligned}
B_{5,4} &= -\int \widetilde{b_2}\partial_2\partial_2\bar{b}\cdot\partial_2\widetilde{u}\,dx \\
&\lesssim \|\partial_2\partial_2\bar{b}\|_{L^2}\|\widetilde{b_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{b_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u}\|_{L^2}^{\frac{1}{2}} \\
&\lesssim \|\partial_2\partial_2\bar{b}\|_{L^2}\|\partial_1\widetilde{b_2}\|_{L^2}^{\frac{1}{2}}\|\partial_2\widetilde{b_2}\|_{L^2}^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u}\|_{L^2} \\
&\lesssim \|b\|_{H^3}\|\nabla\widetilde{b}\|_{L^2}\|\partial_1\nabla\widetilde{u}\|_{L^2} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).
\end{aligned}$$

Collecting the bounds for $B_{5,1}$, $B_{5,2}$, $B_{5,3}$ and $B_{5,4}$,

$$B_5 \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2). \tag{4.8}$$

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Similarly, B_6 and B_8 are bounded by

$$B_6,\ B_8 \lesssim (\|b\|_{H^3} + \|u\|_{H^3}) \times (\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2). \tag{4.9}$$

By lemma 2.1 and the divergence-free condition $\nabla \cdot b = 0$,

$$\begin{aligned}
B_4 + B_7 &= \int \nabla(\widetilde{b\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{u}\,dx + \int \nabla(\widetilde{b\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{b}\,dx \\
&= \int \nabla(b\cdot\nabla\widetilde{b})\cdot\nabla\widetilde{u}\,dx + \int \nabla(b\cdot\nabla\widetilde{u})\cdot\nabla\widetilde{b}\,dx - \int \nabla(\overline{b\cdot\nabla\widetilde{b}})\cdot\nabla\widetilde{u}\,dx \\
&\quad - \int \nabla(\overline{b\cdot\nabla\widetilde{u}})\cdot\nabla\widetilde{b}\,dx \\
&= \int \nabla b\cdot\nabla\widetilde{b}\cdot\nabla\widetilde{u}\,dx + \int \nabla b\cdot\nabla\widetilde{u}\cdot\nabla\widetilde{b}\,dx + \int b\cdot\nabla\nabla\widetilde{b}\cdot\nabla\widetilde{u}\,dx \\
&\quad + \int b\cdot\nabla\nabla\widetilde{u}\cdot\nabla\widetilde{b}\,dx \\
&= \int \nabla b\cdot\nabla\widetilde{b}\cdot\nabla\widetilde{u}\,dx + \int \nabla b\cdot\nabla\widetilde{u}\cdot\nabla\widetilde{b}\,dx \\
&:= B_{4,1} + B_{4,2}.
\end{aligned}$$

We can rewrite $B_{4,1}$ as

$$\begin{aligned}
B_{4,1} &= \int \partial_1 b_1\partial_1\widetilde{b}\cdot\partial_1\widetilde{u}\,dx + \int \partial_1 b_2\partial_2\widetilde{b}\cdot\partial_1\widetilde{u}\,dx \\
&\quad + \int \partial_2 b_1\partial_1\widetilde{b}\cdot\partial_2\widetilde{u}\,dx + \int \partial_2 b_2\partial_2\widetilde{b}\cdot\partial_2\widetilde{u}\,dx \\
&:= B_{4,1,1} + B_{4,1,2} + B_{4,1,3} + B_{4,1,4}.
\end{aligned}$$

By lemma 2.4, lemma 2.3, Hölder's inequality and Young's inequality,

$$\begin{aligned}
B_{4,1,1} &\lesssim \|\partial_1 b_1\|_{L^\infty}\|\partial_1\widetilde{b}\|_{L^2}\|\partial_1\widetilde{u}\|_{L^2} \\
&\lesssim \|\partial_1 b_1\|_{L^2}^{\frac14}(\|\partial_1 b_1\|_{L^2} + \|\partial_1\partial_1 b_1\|_{L^2})^{\frac14}\|\partial_2\partial_1 b_1\|_{L^2}^{\frac14} \\
&\qquad \times (\|\partial_2\partial_1 b_1\|_{L^2} + \|\partial_1\partial_2\partial_1 b_1\|_{L^2})^{\frac14}\|\partial_1\widetilde{u}\|_{L^2} \\
&\lesssim \|b\|_{H^3}\|\partial_1\widetilde{b}\|_{L^2}\|\partial_1\partial_1\widetilde{u}\|_{L^2} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).
\end{aligned}$$

$B_{4,1,2}$, $B_{4,1,3}$ and $B_{4,1,4}$ can be bounded similarly as $B_{4,1,1}$ and

$$B_{4,1,2},\ B_{4,1,3},\ B_{4,1,4} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).$$

Therefore, $B_{4,1}$ is bounded by

$$B_{4,1} \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).$$

Similarly, $B_{4,2}$ obeys the same bound as $B_{4,1}$. Hence,

$$B_4 + B_7 \lesssim \|b\|_{H^3}(\|\nabla\widetilde{b}\|_{L^2}^2 + \|\partial_1\nabla\widetilde{u}\|_{L^2}^2).$$

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Inserting the estimates for B_1 through B_8 in (4.7),

$$\frac{d}{dt}(\|\nabla\widetilde{u}\|_{L^2}^2 + \|\nabla\widetilde{b}\|_{L^2}^2) + (2\nu - C\|(u,b)\|_{H^3})\|\partial_1\nabla\widetilde{u}\|_{L^2}^2 + (2\eta - C\|(u,b)\|_{H^3})\|\nabla\widetilde{b}\|_{L^2} \leqslant 0.$$

Choosing $\varepsilon > 0$ sufficiently small and by theorem 1.1, if $\|u_0\|_{H^3} + \|b_0\|_{H^3} \leqslant \varepsilon$, then $\|u\|_{H^3} + \|b\|_{H^3} \leqslant C\varepsilon$ and

$$2\nu - C\|(u,b)\|_{H^3} \geqslant \nu, \qquad 2\eta - C\|(u,b)\|_{H^3} \geqslant \eta.$$

By lemma 2.3, we obtain the exponential decay result for $\|(\nabla\widetilde{u}(t),\, \nabla\widetilde{b}(t))\|_{L^2}$,

$$\|\nabla\widetilde{u}(t)\|_{L^2} + \|\nabla\widetilde{b}(t)\|_{L^2} \leqslant (\|\nabla u_0\|_{L^2} + \|\nabla b_0\|_{L^2})e^{-C_1 t}, \tag{4.10}$$

where $C_1 = C_1(\nu, \eta) > 0$. Combining the estimates in (4.5) and (4.10), we obtain the desired decay result in theorem 1.2. □

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References

1 A. Alexakis. Two-dimensional behavior of three-dimensional magnetohydrodynamic flow with a strong guiding field. *Phys. Rev. E* **84** (2011), 056330.

2 H. Alfvén. Existence of electromagnetic-hydrodynamic waves. *Nature* **150** (1942), 405–406.

3 C. Bardos, C. Sulem and P. L. Sulem. Longtime dynamics of a conductive fluid in the presence of a strong magnetic field. *Trans. Am. Math. Soc.* **305** (1988), 175–191.

4 D. Biskamp. *Nonlinear Magnetohydrodynamics* (Cambridge: Cambridge University Press, 1993).

5 N. Boardman, H. Lin and J. Wu. Stabilization of a background magnetic field on a 2D magnetohydrodynamic flow. *SIAM J. Math. Anal.* **52** (2020), 5001–5035.

6 P. Burattini, O. Zikanov and B. Knaepen. Decay of magnetohydrodynamic turbulence at low magnetic Reynolds number. *J. Fluid Mech.* **657** (2010), 502–538.

7 Y. Cai and Z. Lei. Global well-posedness of the incompressible magnetohydrodynamics. *Arch. Rational Mech. Anal.*, **228** (2018), 969–993.

8 C. Cao, D. Regmi and J. Wu. The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion. *J. Differ. Eq.* **254** (2013), 2661–2681.

9 C. Cao and J. Wu. Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. *Adv. Math.* **226** (2011), 1803–1822.

10 C. Cao, J. Wu and B. Yuan. The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion. *SIAM J. Math. Anal.* **46** (2014), 588–602.

11 W. Chen, Z. Zhang and J. Zhou. Global well-posedness for the 3-D MHD equations with partial dissipation in periodic domain. *Sci. China Math.* in press.

12 P. A. Davidson. Magnetic damping of jets and vortices. *J. Fluid Mech.* **299** (1995), 153–186.

13 P. A. Davidson. The role of angular momentum in the magnetic damping of turbulence. *J. Fluid Mech.* **336** (1997), 123–150.

14 P. A. Davidson. *An Introduction to Magnetohydrodynamics* (Cambridge, England: Cambridge University Press, 2001).

15 W. Deng and P. Zhang. Large time behavior of solutions to 3-D MHD system with initial data near equilibrium. *Arch. Rational Mech. Anal.* **230** (2018), 1017–1102.

16 B. Dong, Y. Jia, J. Li and J. Wu. Global regularity and time decay for the 2D magnetohydrodynamic equations with fractional dissipation and partial magnetic diffusion. *J. Math. Fluid Mech.* **20** (2018), 1541–1565.

17 B. Dong, J. Li and J. Wu. Global regularity for the 2D MHD equations with partial hyperresistivity. *Intern. Math Res. Notices* **14** (2019), 4261–4280.

18 B. Dong, J. Wu, X. Xu and N. Zhu. Stability and exponential decay for the 2D anisotropic Navier-Stokes equations with horizontal dissipation, submitted for publication.

19 L. Du and D. Zhou. Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion. *SIAM J. Math. Anal.* **47** (2015), 1562–1589.

20 J. Fan, H. Malaikah, S. Monaquel, G. Nakamura and Y. Zhou. Global Cauchy problem of 2D generalized MHD equations. *Monatsh. Math.* **175** (2014), 127–131.

21 C. L. Fefferman, D. S. McCormick, J. C. Robinson and J. L. Rodrigo. Higher order commutator estimates and local existence for the non-resistive MHD equations and related models. *J. Funct. Anal.* **267** (2014), 1035–1056.

22 C. L. Fefferman, D. S. McCormick, J. C. Robinson and J. L. Rodrigo. Local existence for the non-resistive MHD equations in nearly optimal Sobolev spaces. *Arch. Ration. Mech. Anal.* **223** (2017), 677–691.

23 W. Feng F. Hafeez and J. Wu. Influence of a background magnetic field on a 2D magnetohydrodynamic flow. *Nonlinearity* **34** (2021), 2527–2562.

24 B. Gallet, M. Berhanu and N. Mordant. Influence of an external magnetic field on forced turbulence in a swirling flow of liquid metal. *Phys. Fluids* **21** (2009), 085107.

25 B. Gallet and C. R. Doering. Exact two-dimensionalization of low-magnetic-Reynolds-number flows subject to a strong magnetic field. *J. Fluid Mech.* **773** (2015), 154–177.

26 L. He, L. Xu and P. Yu. On global dynamics of three dimensional magnetohydrodynamics: nonlinear stability of Alfvén waves. *Ann. PDE* **4** (2018), 105.

27 X. Hu. Global existence for two dimensional compressible magnetohydrodynamic flows with zero magnetic diffusivity, arXiv: 1405.0274v1 [math.AP] 1 May 2014.

28 X. Hu and F. Lin. Global existence for two dimensional incompressible magnetohydrodynamic flows with zero magnetic diffusivity, arXiv: 1405.0082v1 [math.AP] 1 May 2014.

29 R. Ji and J. Wu. The resistive magnetohydrodynamic equation near an equilibrium. *J. Differ. Eq.* **268** (2020), 1854–1871.

30 Q. Jiu, D. Niu, J. Wu, X. Xu and H. Yu. The 2D magnetohydrodynamic equations with magnetic diffusion. *Nonlinearity* **28** (2015), 3935–3956.

31 C. Li, J. Wu and X. Xu. Smoothing and stabilization effects of magnetic field on electrically conducting fluids. *J. Differ. Eq.* **276** (2021), 368–403.

32 J. Li, W. Tan and Z. Yin. Local existence and uniqueness for the non-resistive MHD equations in homogeneous Besov spaces. *Adv. Math.* **317** (2017), 786–798.

33 F. Lin, L. Xu and P. Zhang. Global small solutions to 2-D incompressible MHD system. *J. Differ. Eq.* **259** (2015), 5440–5485.

34 F. Lin and P. Zhang. Global small solutions to an MHD-type system: the three-dimensional case. *Comm. Pure Appl. Math.* **67** (2014), 531–580.

35 H. Lin and L. Du. Regularity criteria for incompressible magnetohydrodynamics equations in three dimensions. *Nonlinearity* **26** (2013), 219–239.

36 H. Lin, R. Ji, J. Wu and L. Yan. Stability of perturbations near a background magnetic field of the 2D incompressible MHD equations with mixed partial dissipation. *J. Funct. Anal.* **279** (2020), 108519.

37 A. Majda and A. Bertozzi. *Vorticity and Incompressible Flow* (Cambridge, UK: Cambridge University Press, 2002).

38 R. Pan, Y. Zhou and Y. Zhu. Global classical solutions of three dimensional viscous MHD system without magnetic diffusion on periodic boxes. *Arch. Rational Mech. Anal.* **227** (2018), 637–662.

39 E. Priest and T. Forbes. *Magnetic Reconnection, MHD Theory and Applications* (Cambridge: Cambridge University Press, 2000).

40 X. Ren, J. Wu, Z. Xiang and Z. Zhang. Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion. *J. Funct. Anal.* **267** (2014), 503–541.

41 X. Ren, Z. Xiang and Z. Zhang. Global well-posedness for the 2D MHD equations without magnetic diffusion in a strip domain. *Nonlinearity* **29** (2016), 1257–1291.

42 M. E. Schonbek, T. P. Schonbek and E. Süli. Large-time behaviour of solutions to the magnetohydrodynamics equations. *Math. Ann.* **304** (1996), 717–756.

43 M. Sermange and R. Temam. Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.* **36** (1983), 635–664.

44 Z. Tan and Y. Wang. Global well-posedness of an initial-boundary value problem for viscous non-resistive MHD systems. *SIAM J. Math. Anal.* **50** (2018), 1432–1470.

45 T. Tao. *Nonlinear dispersive equations: local and global analysis, CBMS regional conference series in mathematics*, providence, RI: American Mathematical Society, 2006.

46 R. Wan. On the uniqueness for the 2D MHD equations without magnetic diffusion. *Nonlin. Anal. Real World Appl.* **30** (2016), 32–40.

47 D. Wei and Z. Zhang. Global well-posedness of the MHD equations in a homogeneous magnetic field. *Anal. PDE* **10** (2017), 1361–1406.

48 D. Wei and Z. Zhang. Global well-posedness for the 2D MHD equations with magnetic diffusion. *Commun. Math. Res.* **36** (2020), 377–389.

49 J. Wu. The 2D magnetohydrodynamic equations with partial or fractional dissipation, *Lectures on the analysis of nonlinear partial differential equations*, Morningside Lectures on Mathematics, Part 5, MLM5, pp. 283-332, International Press, Somerville, MA, 2018.

50 J. Wu and Y. Wu. Global small solutions to the compressible 2D magnetohydrodynamic system without magnetic diffusion. *Adv. Math.* **310** (2017), 759–888.

51 J. Wu, Y. Wu and X. Xu. Global small solution to the 2D MHD system with a velocity damping term. *SIAM J. Math. Anal.* **47** (2015), 2630–2656.

52 J. Wu and Y. Zhu. Global solutions of 3D incompressible MHD system with mixed partial dissipation and magnetic diffusion near an equilibrium. *Adv. Math.* **377** (2021), 107466.

53 Y. Xiao, Z.-P. Xin and J. Wu. Vanishing viscosity limit for the 3D magneto-hydrodynamic system with a slip boundary condition. *J. Funct. Anal.* **257** (2009), 3375–3394.

54 L. Xu and P. Zhang. Global small solutions to three-dimensional incompressible magneto-hydrodynamical system. *SIAM J. Math. Anal.* **47** (2015), 26–65.

55 K. Yamazaki. On the global well-posedness of N-dimensional generalized MHD system in anisotropic spaces. *Adv. Differ. Eq.* **19** (2014), 201–224.

56 K. Yamazaki. Remarks on the global regularity of the two-dimensional magnetohydrodynamics system with zero dissipation. *Nonlinear Anal.* **94** (2014), 194–205.

57 K. Yamazaki. On the global regularity of two-dimensional generalized magnetohydrodynamics system. *J. Math. Anal. Appl.* **416** (2014), 99–111.

58 K. Yamazaki. Global regularity of logarithmically supercritical MHD system with zero diffusivity. *Appl. Math. Lett.* **29** (2014), 46–51.

59 W. Yang, Q. Jiu and J. Wu. The 3D incompressible Navier-Stokes equations with partial hyperdissipation. *Math. Nach.* **292** (2019), 1823–1836.

60 W. Yang, Q. Jiu and J. Wu. The 3D incompressible magnetohydrodynamic equations with fractional partial dissipation. *J. Differ. Eq.* **266** (2019), 630–652.

61 B. Yuan and J. Zhao. Global regularity of 2D almost resistive MHD equations. *Nonlin. Anal. Real World Appl.* **41** (2018), 53–65.

62 T. Zhang. An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system, (2014), arXiv:1404.5681.

63 T. Zhang. Global solutions to the 2D viscous, non-resistive MHD system with large background magnetic field. *J. Differ. Equ.* **260** (2016), 5450–5480.

64 Y. Zhou and Y. Zhu. Global classical solutions of 2D MHD system with only magnetic diffusion on periodic domain. *J. Math. Phys.* **59** (2018), 081505.