Regularity Criteria on the 2D Anisotropic Magnetic Bénard Equations

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Abstract. In this paper, we study the global regularity issue of two dimensional incompressible magnetic Bénard equations with partial dissipation and magnetic diffusion. It remains open whether the smooth initial data produce solutions that are globally regular in time for all values of the parameters involved in the equations. We present conditional global regularity of the solutions. Moreover, we prove the global regularity for the slightly regularized system.

1 Introduction

This paper aims the global regularity of two dimensional magnetic Bénard equations. The standard two-dimensional incompressible magnetic Bénard equations can be written as

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + b \cdot \nabla b + \theta e_2, \\ b_t + u \cdot \nabla b = \eta \Delta b + b \cdot \nabla u, \\ \partial_t \theta + (u \cdot \nabla)\theta - \kappa \Delta \theta = u \cdot e_2, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \ b(x,y,0) = b_0(x,y), \ \theta(x,y,0) = \theta_0(x,y), \end{cases} \tag{1.1}$$

where $(x,y) \in \mathbb{R}^2$, $t \geq 0$, $u = (u_1(x,y,t), u_2(x,y,t))$ denotes the 2D velocity field, $p = p(x,y,t)$ the pressure, $b = (b_1(x,y,t), b_2(x,y,t))$ the magnetic field, $\theta(x,y,t)$ the temperature, $e_2 = (0,1)^T$ vertical unit vector, and ν, η and κ are nonnegative real parameters.

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A generalized 2D Magnetic Bénard equations can be written as

$$
\begin{cases}
u_t + u \cdot \nabla u = -\nabla p + v_1 u_{xx} + v_2 u_{yy} + b \cdot \nabla b + \theta e_2, \\
b_t + u \cdot \nabla b = \eta_1 b_{xx} + \eta_2 b_{yy} + b \cdot \nabla u, \\
\partial_t \theta + (u \cdot \nabla)\theta - \kappa_1 \partial_{xx}\theta - \kappa_2 \partial_{yy}\theta = u_2, \\
\nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\
u(x,y,0) = u_0(x,y), \ b(x,y,0) = b_0(x,y), \ \theta(x,y,0) = \theta_0(x,y).
\end{cases}
\tag{1.2}
$$

If $v_1 = v_2 = v$ and $\eta_1 = \eta_2 = \eta$, and $\kappa_1 = \kappa_2 = \kappa$, then (1.2) reduces to the standard magnetic Bénard equations (1.1). This generalization is capable of modeling the motion of anisotropic fluids for which the diffusion properties in different directions are different.

In the absence of θ, the magnetic Bénard equation reduces to magneto-hydrodynamic (MHD) equation. When all four parameters v_1, v_2, η_1, and η_2 are positive, the global regularity of the classical solution to 2D MHD equations has been established, see, e.g., [7], [19]. On the other hand, it remains a remarkable open problem whether classical solutions of the two-dimensional inviscid MHD equations, with all four parameters equal to zero, preserve their regularity for all time or have finite time blowup. Many attempts have been made but there are no any satisfactory results concerning the regularity of the solution. When $v_1 > 0$, $v_2 = 0$, $\eta_1 = 0$ and $\eta_2 > 0$ or when $v_1 = 0$, $v_2 > 0$, $\eta_1 > 0$ and $\eta_2 = 0$, the global regularity was established by Cao and Wu in [2]. Cao, Regmi, and Wu studied two dimensional MHD equations with horizontal dissipation and horizontal diffusion in [1]. They proved that any possible blow-up can be controlled by the L^∞-norm of the horizontal components.

There are numerous papers related to two dimensional MHD equations [1–8, 16, 20, 23, 25] and references therein, however only few papers are available related to magnetic Bénard equations. Y. Zhou et al. in [32] obtained the global regularity results related to the 2D magnetic Bénard problem with zero thermal conductivity. The authors used energy estimates as well as a well known property of Hardy space and Bounded Mean Value Oscillation (BMO) to prove the global regularity. Very recently, J. Cheng and L. Du in [6] proved the global well-posedness of the 2D Magnetic Bénard equations with mixed partial viscosity which included vertical or horizontal magnetic diffusion but no thermal diffusivity. The authors also obtained the global regularity as well as some conditional regularity of strong solutions of the problem with mixed partial viscosity, thus extending the existing result of the problem with the full dissipation. Likewise, the global regularity of generalized magnetic Bénard problem was studied by Y. Yamazaki in [28] by extending the existing results on Boussinesq equation and magneto-hydrodynamic equations. The author studied the problem with fractional Laplacian and logarithmic super criticality. The author showed that when the diffusive term has a full Laplacian, then a sufficiently smooth initial data evolves into a smooth solution under certain conditions. The author also presented additional global regularity criteria for the velocity field, magnetic field and the temperature field.

This paper is devoted to the case when $\nu_1 > 0$, $\nu_2 = 0$, $\eta_1 > 0$, $\eta_2 = 0$. $\kappa_1 > 0$, and $\kappa_2 = 0$, namely

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + u_{xx} + b \cdot \nabla b + \theta e_2, \\ b_t + u \cdot \nabla b = b_{xx} + b \cdot \nabla u, \\ \partial_t \theta + (u \cdot \nabla)\theta - \partial_{xx}\theta = u_2, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \ b(x,y,0) = b_0(x,y), \end{cases} \tag{1.3}$$

where we have set $\nu_1 = \eta_1 = \kappa_1 = 1$. This paper presents conditioned global regularity. More precisely, we prove the following theorem.

Theorem 1.1. *Assume that* $(u_0, b_0, \theta_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then, (1.3) has a unique classical solution* (u, b, θ) *satisfying, for any* $T > 0$,

$$u, b, \theta, \partial_x u, \partial_x b, \partial_x \theta \in L^\infty([0,T]; H^2(\mathbb{R}^2))$$

provided $\int_0^T \|u_1, b_1\|_{BMO}^2 dt < \infty$.

This result is also true for the following system.

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + u_{xx} + b \cdot \nabla b + \theta e_2, \\ b_t + u \cdot \nabla b = b_{xx} + b \cdot \nabla u, \\ \partial_t \theta + (u \cdot \nabla)\theta - \partial_{yy}\theta = u_2, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \ b(x,y,0) = b_0(x,y). \end{cases} \tag{1.4}$$

If we follow the proof of Theorem 1.1, we can further prove the following theorem.

Theorem 1.2. *Consider*

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + u_{yy} + b \cdot \nabla b + \theta e_2, \\ b_t + u \cdot \nabla b = b_{yy} + b \cdot \nabla u, \\ \partial_t \theta + (u \cdot \nabla)\theta - \partial_{xx}\theta = u_2, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \quad b(x,y,0) = b_0(x,y). \end{cases} \tag{1.5}$$

Assume that $(u_0, b_0, \theta_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then, (1.3) has a unique classical solution* (u, b, θ) *satisfying, for any* $T > 0$,

$$u, b, \theta, \partial_x u, \partial_x b, \partial_x \theta \in L^\infty([0,T]; H^2(\mathbb{R}^2))$$

provided $\int_0^T \|u_2, b_2\|_{BMO}^2 dt < \infty$.

Remark 1.1.   (1) In [6], authors presented conditional regularity results for the system (1.3) and the results are different than we have presented here.

(2) Our results improves the previously known results related to MHD equations with horizontal dissipation and magnetic diffusion in [1].

We furthermore consider slightly regularized version of (1.3), namely either for any $\epsilon, \delta > 0$

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu_2 u_{xx} + \epsilon(-\Delta)^\delta u + b \cdot \nabla b + \theta e_2, \\ b_t + u \cdot \nabla b = \eta_2 b_{xx} + \epsilon(-\Delta)^\delta b + b \cdot \nabla u, \\ \partial_t \theta + (u \cdot \nabla)\theta - \partial_{xx}\theta = u_2, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \quad b(x,y,0) = b_0(x,y), \end{cases} \tag{1.6}$$

or

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu_2 u_{xx} + \epsilon(-\Delta)^\delta u + b \cdot \nabla b + \theta e_2, \\ b_t + u \cdot \nabla b = \eta_2 b_{xx} + \epsilon(-\Delta)^\delta b + b \cdot \nabla u, \\ \partial_t \theta + (u \cdot \nabla)\theta - \partial_{yy}\theta = u_2, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y), \quad b(x,y,0) = b_0(x,y). \end{cases} \tag{1.7}$$

We prove the following theorem.

Theorem 1.3. *Assume that* $(u_0, b_0, \theta_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *Then, (1.6) or (1.7) has a unique classical solution* (u, b, θ) *satisfying, for any* $T > 0$,

$$u, b, \theta, \partial_x u, \partial_x b, \partial_x \theta \in L^\infty([0,T]; H^2(\mathbb{R}^2)).$$

Similar result is true if we consider the vertical dissipation and vertical magnetic diffusion.

The general approach to establish the global existence and regularity consists of two main steps. The first step is local existence and uniqueness and the second step is global *a priori bounds*. For this type of system, local existence follows from standard approach (we omit here). We only concentrate to obtain the global bound. The main difficulty to obtain global bound for aforementioned system is H^1 bound. In H^1 bound, we encounter the terms $\int j \partial_x b_1 \partial_y u_1$ and $\int j \partial_x u_1 \partial_y b_1$. Unfortunately, we do not know how to bound the other two terms in order to close the inequality due to insufficient vertical dissipation and vertical magnetic diffusion. This is where the direct energy method breaks down and the global regularity problem becomes very hard.

The rest of this paper is divided into three sections. The last two sections are devoted to the proof for each of the theorems stated above.

2   Preliminaries

To simplify the notation, we will write $\|f\|_2$ for $\|f\|_{L^2}$, $\int f$ for $\int_{\mathbb{R}^2} f \, dx dy$ and write $\frac{\partial}{\partial x} f$, $\partial_x f$ or f_x as the first partial derivative, and $\frac{\partial^2}{\partial x^2} f$ or $\partial_{xx} f$ as the second partial throughout the rest of this paper. BMO represents the bounded mean oscillation.

The following anisotropic type Sobolev inequality will be frequently used. Its proof can be found in [2].

Lemma 2.1. *If $f,g,h,\partial_y g,\partial_x h \in L^2(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} |fgh| \, dxdy \leq C \|f\|_2 \|g\|_2^{\frac{1}{2}} \|\partial_y g\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|\partial_x h\|_2^{\frac{1}{2}}, \tag{2.1}$$

where C is a constant.

The following simple fact on the boundedness of Riesz transforms will also be used.

Lemma 2.2. *Let f be divergence-free vector field such that $\nabla f \in L^p$ for $p \in (1,\infty)$. Then there exists a pure constant $C > 0$ (independent of p) such that*

$$\|\nabla f\|_{L^p} \leq \frac{Cp^2}{p-1} \|\nabla \times f\|_{L^p}.$$

3 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. As explained in the introduction, it suffices to establish the global *a priori* bound for the solution in H^2. For the sake of clarity, we divide this process into two subsections. The first subsection proves the global H^1-bound while the second proves the global H^2-bound.

3.1 H^1-Bound

We first state the global L^2-bound.

Lemma 3.1. *Assume that (u_0,b_0,θ_0) satisfies the condition stated in Theorem 1.1. Let (u,b,θ) be the corresponding solution of (1.3). Then, (u,b,θ) obeys the following global L^2-bound,*

$$\|u(t),b(t),\theta(t)\|_{L^2}^2 + 2\int_0^t \|\partial_x u,\partial_x b,\partial_x \theta\|_{L^2}^2 \, d\tau \leq C(\|(u_0,b_0,\theta_0)\|_2^2)$$

for any $t \geq 0$.

We can easily prove the global L^2 bound. Taking the L^2-inner product of (u,b,θ) with (1.3), respectively, and adding together yields

$$\frac{1}{2}\frac{d}{dt}\left(\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\theta(t)\|_{L^2}^2\right) + \|\partial_x u(\tau)\|_{L^2}^2 + \|\partial_x b(\tau)\|_{L^2}^2 + \|\partial_x \theta(\tau)\|_{L^2}^2$$
$$= \int \theta e_2 \cdot u + \int u_2 \theta \leq C\|u\|_{L^2}\|\theta\|_{L^2}.$$

Combining together, we easily obtain the global L^2 bound.

$$\|u(t),b(t),\theta(t)\|_{L^2}^2 + 2\int_0^t \|\partial_x u(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|\partial_x b(\tau)\|_{L^2}^2 d\tau + 2\int \|\partial_x\theta\|_2^2 d\tau \leq C,$$

for any $0 < t \leq T$, where C depends only on the initial data. We next prove the global H^1-bound for u,b and θ. More precisely, we prove the following result.

Proposition 3.1. Assume that (u_0,b_0,θ_0) satisfies the condition stated in Theorem 1.1. Let (u,b,θ) be the corresponding solution of (1.3). Then (u,b,θ) satisfies, for any $T > 0$,

$$u,b,\theta \in C([0,T];H^1(\mathbb{R}^2)). \tag{3.1}$$

Consider the equation for $\omega = \nabla \times u$ and $j = \nabla \times b$ to estimate H^1,

$$\begin{cases} \omega_t + u\cdot\nabla\omega = \omega_{xx} + b\cdot\nabla j + \partial_x\theta, \\ j_t + u\cdot\nabla j = j_{xx} + b\cdot\nabla\omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1). \end{cases} \tag{3.2}$$

We then obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \|\omega_x\|_{L^2}^2 + \|j_x\|_{L^2}^2$$
$$= 2\int j\left(\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1)\right) + \int \partial_x\theta\omega$$
$$=: I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.3}$$

For notational convenience, we will omit $dxdy$ from the spatial integral. The first term can be bounded by using Lemma 2.1

$$I_1 = \left|2\int \partial_x b_1 \partial_x u_2 j\right| \leq C\|\partial_x u_2\|_{L^2}\|\partial_x b_1\|_{L^2}^{\frac{1}{2}}\|\partial_x\partial_y b_1\|_{L^2}^{\frac{1}{2}}\|j\|_{L^2}^{\frac{1}{2}}\|\partial_x j\|_{L^2}^{\frac{1}{2}}.$$

Applying Young's inequality and the simple fact that

$$\|\partial_x b_1\|_{L^2} \leq \|j\|_{L^2}, \qquad \|\partial_x\partial_y b_1\|_{L^2} \leq \|\partial_x j\|_{L^2},$$

we have

$$I_1 \leq \frac{1}{8}\|\partial_x j\|_{L^2}^2 + C\|\partial_x u_2\|_{L^2}^2\|j\|_{L^2}^2.$$

Similarly,

$$I_3 = \left|2\int \partial_x u_1 \partial_x b_2 j\right| \leq \frac{1}{8}\|\partial_x j\|_{L^2}^2 + C\|\partial_x u_1\|_{L^2}^2\|j\|_{L^2}^2.$$

The terms I_2 and I_4 have to be handled differently. By integration by parts,

$$I_2 = 2\int \partial_x b_1 \partial_y u_1 j = -2\int b_1 \partial_x\partial_y u_1 j - 2\int b_1 \partial_y u_1 \partial_x j \tag{3.4}$$

which gives that

$$I_2 \le \frac{1}{8}(\|\omega_x\|_2^2 + \|j_x\|_2^2) + C\|b_1\|_{BMO}^2(\|j\|_2^2 + \|\omega\|_2^2).$$

Similarly

$$I_4 = -2\int \partial_x u_1 \partial_y b_1 j = -2\int \partial_x u_1 \partial_y b_1 \partial_x b_2 - 2\int \partial_x u_1 \partial_y b_1 \partial_y b_1 =: I_{41} + I_{42}.$$

The last two terms admit

$$I_{41} \le \frac{1}{8}\|\omega_x\|_2^2 + \frac{1}{8}\|j_x\|_2^2 + C\|b\|_2^2\|\partial_x b\|_2^2(\|\omega\|_2^2 + \|j\|_2^2),$$

$$I_{42} = -4\int u_1 \partial_y b_1 \partial_{xy} b_1 \le \frac{1}{8}\|\partial_x j\|_2^2 + \frac{1}{8}\|u_1\|_{BMO}^2\|j\|_2^2.$$

Furthermore,

$$I_5 \le \left|\int \partial_x \theta \omega\right| \le \|\theta\|_2^2 + \|\omega_x\|_2^2.$$

After combining all inequalities

$$\frac{1}{2}\frac{d}{dt}(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2) + \|\partial_x \omega\|_{L^2}^2 + \|\partial_x j\|_{L^2}^2 \le \frac{1}{2}\|\partial_x \omega\|_2^2 + \frac{1}{2}\|\partial_x j\|_2^2$$
$$+ C(\|u_1\|_{BMO}^2 + \|b_1\|_{BMO}^2 + \|\partial_x b\|_2^2 + \|\theta\|_2^2)(\|j\|_2^2 + \|\omega\|_2^2).$$

Gronwall's lemma yields,

$$\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \int_0^t \|\partial_x \omega\|_{L^2}^2 + \int_0^t \|\partial_x j\|_{L^2}^2 \le Ce^{C\int_0^t(\|u_1\|_{BMO}^2 + \|b_1\|_{BMO}^2 + \|\partial_x b\|_2^2)}.$$

Now we need to know the H^1 bound for θ. Taking inner products of the third equation in (1.3) with $\Delta\theta$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_2^2 + \kappa\|\nabla\partial_x\theta\|_2^2 = -\int \nabla u \cdot \nabla\theta \cdot \nabla\theta + (u\cdot\nabla)\nabla\theta \cdot \nabla\theta - \nabla u_2 \cdot \nabla\theta, \qquad (3.5)$$

$$-\int \nabla u \cdot \nabla\theta \cdot \nabla\theta = -\int \left(\partial_x u_1(\partial_x\theta)^2 + \partial_x u_2 \partial_x\theta\partial_y\theta + \partial_y u_1 \partial_x\theta\partial_y\theta + \partial_y u_2(\partial_y\theta)^2\right)$$
$$= J_1 + J_2 + J_3 + J_4. \qquad (3.6)$$

By the divergence free condition $\int (u\cdot\nabla)\nabla\theta \cdot \nabla\theta = 0$ and

$$\int \nabla u_2 \cdot \nabla\theta = \int \partial_x u_2 \theta_x + \int \partial_y u_2 \theta_y =: J_5 + J_6,$$

we have

$$J_1 = -2\int u_2 \partial_x\theta \partial_{xy}\theta \leq C\|\partial_{xy}\theta\|_2^{\frac{3}{2}}\|u_2\|_2^{\frac{1}{2}}\|\partial_x\theta\|_2^{\frac{1}{2}} \leq \frac{1}{48}\|\nabla\partial_x\theta\|_2^2 + C\|u_2\|_2^2\|\partial_x\theta\|_2^2,$$

$$J_2 = \int \partial_x u_2 \partial_x\theta \partial_y\theta \leq C\|\partial_x u_2\|_2\|\partial_x\theta\|_2^{\frac{1}{2}}\|\partial_y\theta\|_2^{\frac{1}{2}}\|\partial_{xy}\theta\|_2 \leq \frac{1}{48}\|\nabla\partial_x\theta\|_2^2 + C\|\partial_x u_2\|_2^2\|\nabla\theta\|_2^2.$$

Similarly,

$$J_3 = \int \partial_y u_1 \partial_x\theta \partial_y\theta \leq C\|\partial_y u_1\|_2\|\partial_x\theta\|_2^{\frac{1}{2}}\|\partial_y\theta\|_2^{\frac{1}{2}}\|\partial_{xy}\theta\|_2$$
$$\leq \frac{1}{48}\|\nabla\partial_x\theta\|_2^2 + C\|\partial_y u\|_2^2\|\nabla\theta\|_2^2,$$

$$J_4 = -2\int u_1 \partial_{xy}\theta \partial_y\theta \leq \frac{1}{48}\|\nabla\partial_x\theta\|_2^2 + C\|u\|_2^2\|\nabla u\|_2^2\|\nabla\theta\|_2^2,$$

$$J_5 = \int \partial_x u_2 \theta_x \leq \|\partial_x u_2\|_2\|\partial_x\theta\|_2, \tag{3.7}$$

$$J_6 = \int \partial_y u_2 \theta_y \leq \|\partial_y u_2\|_2\|\partial_y\theta\|_2.$$

Combining all inequalities together with Gronwall's lemma yields,

$$\|\nabla\theta(t)\|_2^2 + \int_0^t \|\nabla\partial_x\theta\|_2^2 \leq C$$

for any $t \leq T$. This completes the global H^1 bound for (u,b,θ).

3.2 H^2-bound

Taking the inner with of (3.2) with $(\Delta\omega, \Delta j)$ yields

$$\frac{1}{2}\frac{d}{dt}\|\nabla\omega\|_2^2 + \|\nabla\omega_x\|_2^2 = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega \, dxdy + \int \nabla\omega\cdot\nabla b\cdot\nabla j \, dxdy$$
$$+ \int b\cdot\nabla(\nabla j)\cdot\nabla\omega \, dxdy + \int \partial_x\theta\Delta\omega \, dxdy,$$

$$\frac{1}{2}\frac{d}{dt}\|\nabla j\|_2^2 + \|\nabla j_x\|_2^2 = -\int \nabla j\cdot\nabla u\cdot\nabla j \, dxdy + \int \nabla j\cdot\nabla b\cdot\nabla\omega \, dxdy \int b\cdot\nabla(\nabla\omega)\cdot\nabla j \, dxdy$$
$$+ 2\int \nabla[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]\cdot\nabla j \, dxdy - 2\int \nabla[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\cdot\nabla j \, dxdy.$$

Adding above equations and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2 + \|\Delta\theta\|_2^2) + \|\nabla\omega_x\|_2^2 + \|\nabla j_x\|_2^2 + \|\nabla\Delta\theta\|_2^2 =: \sum_{i=1}^8 K_i,$$

where

$$K_1 = -\int \nabla \omega \cdot \nabla u \cdot \nabla \omega \, dxdy, \quad K_2 = -\int \nabla j \cdot \nabla u \cdot \nabla j \, dxdy,$$

$$K_3 = 2\int \nabla \omega \cdot \nabla b \cdot \nabla j \, dxdy, \quad K_4 = 2\int \nabla [\partial_x b_1 (\partial_x u_2 + \partial_y u_1)] \cdot \nabla j \, dxdy,$$

$$K_5 = -2\int \nabla [\partial_x u_1 (\partial_x b_2 + \partial_y b_1)] \cdot \nabla j \, dxdy, \quad K_6 = \int \partial_x \theta \Delta \omega \, dxdy,$$

$$K_7 = \int \Delta (u \cdot \nabla) \theta \Delta \theta \, dxdy, \quad K_8 = \int \Delta u_2 \Delta \theta \, dxdy.$$

Observe that

$$
\begin{aligned}
K_1 &= \int (\nabla \omega \cdot \nabla u \cdot \nabla \omega) \, dxdy \\
&= \int (\partial_x u_1 \omega_x^2 + \partial_x u_2 \omega_x \omega_y + \partial_y u_1 \omega_x \omega_y + \partial_y u_2 \omega_y^2) \, dxdy \\
&=: K_{11} + K_{12} + K_{13} + K_{14}.
\end{aligned}
$$

By Lemma 2.1, we have

$$K_{11} \leq C \|\partial_x u_1\|_2 \|\omega_x\|_2^{\frac{1}{2}} \|\omega_{xx}\|_2^{\frac{1}{2}} \|\omega_x\|_2^{\frac{1}{2}} \|\omega_{xy}\|_2^{\frac{1}{2}} \leq \frac{1}{48} \|\nabla \omega_x\|_2^2 + C \|\omega\|_2^2 \|\nabla \omega\|_2^2.$$

Similarly, we obtain

$$
\begin{aligned}
K_{12} &\leq C \|\partial_x u_2\|_2 \|\omega_x\|_2^{\frac{1}{2}} \|\omega_{xy}\|_2^{\frac{1}{2}} \|\omega_y\|_2^{\frac{1}{2}} \|\omega_{xy}\|_2^{\frac{1}{2}} \\
&\leq C \|\omega\|_2 \|\nabla \omega\|_2 \|\nabla \omega_x\|_2 \leq \frac{1}{48} \|\nabla \omega_x\|_2^2 + \|\omega\|_2^2 \|\nabla \omega\|_2^2.
\end{aligned}
$$

Furthermore,

$$K_{13} \leq \frac{1}{48} \|\nabla \omega_x\|_2^2 + \|\omega\|_2^2 \|\nabla \omega\|_2^2, \quad K_{14} \leq \frac{1}{48} \|\nabla \omega_x\|_2^2 + C \|\omega\|_2^{\frac{2}{3}} \|\omega_x\|_2^{\frac{2}{3}} \|\nabla \omega\|_2^2.$$

Similarly,

$$
\begin{aligned}
K_2 &= -\int \nabla j \cdot \nabla u \cdot \nabla j \, dxdy \\
&= \int \partial_x u_1 j_x^2 + \partial_y u_1 j_x j_y + \partial_y u_2 j_y^2 + \partial_x u_2 j_x j_y =: K_{21} + K_{22} + K_{23} + K_{24}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
K_{21} &\leq C \|j_x\|_2 \|\partial_x u_1\|_2^{\frac{1}{2}} \|\partial_{xy} u_1\|_2^{\frac{1}{2}} \|j_x\|_2^{\frac{1}{2}} \|j_{xx}\|_2^{\frac{1}{2}} \\
&\leq C \|\omega\|_2^{\frac{1}{2}} \|\omega_x\|_2^{\frac{1}{2}} \|\nabla j\|_2^{\frac{3}{2}} \|\nabla j_x\|_2^{\frac{1}{2}} \leq \frac{1}{48} \|\nabla j_x\|_2^2 + C \|\omega\|_2^{\frac{2}{3}} \|\omega_x\|_2^{\frac{2}{3}} \|\nabla j\|_2^2,
\end{aligned}
$$

$$K_{22} \leq \frac{1}{48} \|\nabla j_x\|_2^2 + C \|\omega\|_2^{\frac{2}{3}} \|\omega_x\|_2^{\frac{2}{3}} \|\nabla j\|_2^2, \quad K_{23} \leq \frac{1}{48} \|\nabla j_x\|_2^2 + \|\omega\|_2^{\frac{2}{3}} \|\omega_x\|_2^{\frac{2}{3}} \|\nabla j\|_2^2,$$

$$K_{24} \leq \frac{1}{48} \|\nabla j_x\|_2^2 + C \|\omega\|_2^2 \|\nabla j\|_2^2.$$

On the other hand,

$$K_3 = \int \omega_x \partial_x b_1 j_x + \omega_x \partial_x b_2 j_y + \omega_y \partial_y b_1 j_x + \omega_y \partial_y b_2 j_y =: K_{31} + K_{32} + K_{33} + K_{34}.$$

Observe that

$$K_{31} = \int \omega_x \partial_x b_1 j_x \leq C \|\partial_x b_1\|_2 \|\omega_x\|_2^{\frac{1}{2}} \|\omega_{xy}\|_2^{\frac{1}{2}} \|j_x\|_2^{\frac{1}{2}} \|j_{xx}\|_2^{\frac{1}{2}}$$

$$\leq C \|j\|_2 \|\nabla \omega\|_2^{\frac{1}{2}} \|\nabla \omega_x\|_2^{\frac{1}{2}} \|\nabla j\|_2^{\frac{1}{2}} \|\nabla j_x\|_2^{\frac{1}{2}}$$

$$\leq \frac{1}{48} \|\nabla \omega_x\|_2^2 + \frac{1}{48} \|\nabla j_x\|_2^2 + C \|j\|_2^2 (\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2).$$

Similarly,

$$K_{32} \leq \frac{1}{48} \|\nabla \omega_x\|_2^2 + \frac{1}{48} \|\nabla j_x\|_2^2 + C \|j\|_2^2 (\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2),$$

$$K_{33} \leq \frac{1}{48} (\|\nabla w_x\|_2^2 + \|\nabla j_x\|_2^2) + C \|j\|_2^2 (\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2),$$

$$K_{34} \leq \frac{1}{48} \|\nabla j_x\|_2^2 + C \|j_x\|_2^2 \|\nabla j\|_2^2 + C \|j\|_2 \|\nabla \omega\|_2^2.$$

Note that

$$K_4 = 2 \int \nabla [\partial_x b_1 (\partial_x u_2 + \partial_y u_1)] \cdot \nabla j \, dx dy$$

$$= 2 \int \partial_x [\partial_x b_1 (\partial_x u_2 + \partial_y u_1)] j_x + \partial_y [\partial_x b_1 (\partial_x u_2 + \partial_y u_1)] j_y \, dx dy$$

$$=: K_{41} + K_{42}.$$

Observe that

$$K_{41} = -2 \int \partial_x b_1 (\partial_x u_2 + \partial_y u_1) j_{xx}$$

$$\leq C \left(\|\partial_x b_1\|_2^{\frac{1}{2}} \|\partial_{xy} b_1\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_{xy} u_2\|_2^{\frac{1}{2}} + C \|\partial_x b_1\|_2^{\frac{1}{2}} \|\partial_{xy} b_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_{xy} u_1\|_2^{\frac{1}{2}} \right) \|j_{xx}\|_2$$

$$\leq C \|j\|_2^{\frac{1}{2}} \|\nabla j\|_2^{\frac{1}{2}} \|\omega\|_2^{\frac{1}{2}} \|\omega_y\|_2^{\frac{1}{2}} \|\nabla j_x\|_2$$

$$\leq \frac{1}{48} \|\nabla j_x\|_2^2 + C \|\omega\|_2 \|j\|_2 (\|\nabla \omega\|_2^2 + \|\nabla j\|_2^2).$$

We further split K_{42} into four parts

$$K_{42} = 2 \int (\partial_{xy} b_1 \partial_x u_2 + \partial_x b_1 \partial_{xy} u_2 + \partial_{xy} b_1 \partial_y u_1 + \partial_x b_1 \partial_{yy} u_1) j_y \, dx dy$$

$$=: K_{421} + K_{422} + K_{423} + K_{424}.$$

We also provide estimates to each of the four terms:

$$
\begin{aligned}
K_{421} &\leq C\|\partial_{xy}b_1\|_2\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xy}u_2\|_2^{\frac{1}{2}}\|j_y\|_2^{\frac{1}{2}}\|j_{xy}\|_2^{\frac{1}{2}} \\
&\leq C\|j_x\|_2\|\omega\|_2^{\frac{1}{2}}\|\omega_x\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla j_x\|_2^{\frac{1}{2}} \\
&\leq C\|\nabla j\|_2^{\frac{3}{2}}\|_2\|\omega\|_2^{\frac{1}{2}}\|\nabla j_x\|_2^{\frac{1}{2}}\|\omega_x\|_2^{\frac{1}{2}} \\
&\leq \frac{1}{48}\|\nabla j_x\|_2^2+\|\omega\|_2^{\frac{2}{3}}\|\omega_x\|_2^{\frac{2}{3}}\|\nabla j\|_2^2.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
K_{422} &\leq \frac{1}{48}\|\nabla j_x\|_2^2+C\|j_x\|_2^2\|\nabla j\|_2^2+C\|j\|_2\|\nabla\omega\|_2^2, \\
K_{423} &\leq \frac{1}{48}\|\nabla j_x\|_2^2+C\|\omega\|_2\|\nabla j\|_2^2+C\|j_x\|_2^2\|\nabla\omega\|_2^2, \\
K_{424} &\leq \frac{1}{48}\|\nabla\omega_x\|_2^2+C\|j_x\|_2^2\|\nabla\omega\|_2^2+C\|j\|_2\|\nabla j\|_2^2.
\end{aligned}
$$

Two more terms remain to be estimated. First, we work on K_5:

$$
\begin{aligned}
K_5 &= -2\int \nabla[\partial_x u_1(\partial_x b_2+\partial_y b_1)]\cdot\nabla j\,dxdy \\
&= -2\int \partial_x[\partial_x u_1(\partial_x b_2+\partial_y b_1)]j_x+\partial_y[(\partial_x u_1(\partial_x b_2+\partial_y b_1)]j_y\,dxdy \\
&=: K_{51}+K_{52}.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
K_{51} &\leq C\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_x b_2\|_2^{\frac{1}{2}}\|\partial_{xx}b_2\|_2^{\frac{1}{2}}\|j_{xx}\|_2 \\
&\quad +C\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy}b_1\|_2^{\frac{1}{2}}\|j_{xx}\|_2^{\frac{1}{2}} \\
&\leq C\|\omega\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla j_x\|_2 \\
&\leq \frac{1}{48}\|\nabla j_x\|_2^2+C\|\omega\|_2\|j\|_2(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2), \\
K_{52} &= -2\int(\partial_{xy}u_1\partial_x b_2+\partial_x u_1\partial_{xy}b_2+\partial_{xy}u_1\partial_y b_1+\partial_x u_1\partial_{yy}b_1)j_y\,dxdy \\
&=: K_{521}+K_{522}+K_{523}+K_{524}.
\end{aligned}
$$

Note that K_{521} and K_{522} can be bounded as K_{422} and K_{421}. We only need to bound K_{523} and K_{524}.

$$K_{523} = -2\int \partial_{xy}u_1\partial_y b_1 j_y dxdy$$

$$\leq C\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{xyy}u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy}b_1\|_2^{\frac{1}{2}}\|j_y\|_2$$

$$\leq C\|\nabla\omega_x\|_2^{\frac{1}{2}}\|j_x\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2$$

$$\leq \frac{1}{48}\|\nabla\omega_x\|_2^2 + \|j_x\|_2^2\|\nabla\omega\|_2^2 + \|j\|_2\|\nabla j\|_2^2.$$

Finally, note that

$$K_6 \leq \|\nabla\partial_x\theta\|_2^2\|\nabla\omega\|_2 \leq \frac{1}{48}\|\nabla\partial_x\theta\|_2^2 + C\|\nabla\omega\|_2^2.$$

After combining all inequalities, together with the Gronwall's lemma yields

$$\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2 + \int_0^t(\|\nabla\partial_x\omega(\tau)\|_2^2 + \|\nabla\partial_x j(\tau)\|_2^2)d\tau \leq C. \tag{3.8}$$

For the global bound for $\|\Delta\theta\|_2$, applying Δ to the third equation in (1.3) with $\Delta\theta$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\theta\|_2^2 + \|\Delta\partial_x\theta\|_2^2 = \int \Delta(u\cdot\nabla)\theta\Delta\theta dxdy + \int \Delta u_2\Delta\theta dxdy.$$

By Lemma 2.1

$$\int \Delta(u\cdot\nabla)\theta\Delta\theta dxdy = \int \Delta u_1\theta_x\Delta\theta + 2\nabla u_1\cdot\nabla\theta_x\Delta\theta + \Delta u_2\theta_y\Delta\theta + 2\nabla u_2\cdot\nabla\theta_y\Delta\theta.$$

Now we observe that

$$\left|\int \Delta u_1\theta_x\Delta\theta\right| \leq \|\Delta u_1\|_2\|\theta_x\|_2^{\frac{1}{2}}\|\theta_{xy}\|_2^{\frac{1}{2}}\|\Delta\theta\|_2^{\frac{1}{2}}\|\Delta\theta_x\|_2^{\frac{1}{2}}$$

$$\leq \frac{1}{48}\|\Delta\theta_x\|_2^2 + C\|\theta_x\|_2^{\frac{2}{3}}\|\theta_{xy}\|_2^{\frac{2}{3}}\|\Delta u_1\|_2^{\frac{4}{3}}\|\Delta\theta\|_2^{\frac{2}{3}}$$

$$\leq \frac{1}{48}\|\Delta\theta_x\|_2^2 + C\|\theta_x\|_2^{\frac{2}{3}}\|\theta_{xy}\|_2^{\frac{2}{3}}\left(\|\Delta u_1\|_2^2 + \|\Delta\theta\|_2^2\right).$$

Similarly

$$\left|\int 2\nabla u_1\cdot\nabla\theta_x\Delta\theta\right| \leq \frac{1}{48}\|\Delta\theta_x\|_2^2 + C\|\nabla u_1\|_2^{\frac{2}{3}}\|\nabla\partial_x u_1\|_2^{\frac{2}{3}}\|\Delta\theta\|_2^2,$$

$$\left|\int \Delta u_2\theta_y\Delta\theta\right| \leq \frac{1}{48}\|\Delta\partial_x u_2\|_2^2 + \frac{1}{48}\|\Delta\theta_x\|_2^2 + C\|\theta_y\|_2^2\left(\|\Delta u_2\|_2^2 + \|\Delta\theta\|_2^2\right),$$

$$\left|\int 2\nabla u_2\cdot\nabla\theta_y\Delta\theta\right| \leq \frac{1}{48}\|\Delta\theta_x\|_2^2 + C\|\nabla u_2\|_2^{\frac{2}{3}}\|\nabla\partial_x u_1\|_2^{\frac{2}{3}}\|\Delta\theta\|_2^2.$$

Finally,

$$\left| \int \Delta u_2 \Delta \theta \, dx dy \right| \leq \frac{1}{48} \|\Delta u_2\|_2^2 + C \|\Delta \theta\|_2^2.$$

Collecting all inequalities and applying Gronwall's lemma, we obtain

$$\|\Delta \theta\|_2^2 + \int_0^t \|\Delta \partial_x \theta\|_2^2 \leq C$$

for any $t \leq T$. This completes the proof of Theorem 1.1.   □

4   Global regularity of slightly regularized system

This section establishes that (1.6) possesses global regular solutions if the initial data are sufficiently smooth. More precisely, we prove Theorem 1.3. The difficult part to show the global regularity is obtaining global H^1 bound since global H^2-bound is similar to Theorem 1.1.

To obtain the global bound for the H^1-norm, we take advantage of the vorticity formulation. Taking the curl of (1.6), we find that $\omega = \nabla \times u$ and $j = \nabla \times b$ satisfy

$$\begin{cases} \omega_t + u \cdot \nabla \omega + \epsilon(-\Delta)^\delta \omega = b \cdot \nabla j + \omega_{xx} + \partial_x \theta, \\ j_t + u \cdot \nabla j + \epsilon(-\Delta)^\delta j = b \cdot \nabla \omega + j_{xx} + 2\partial_x b_1(\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1(\partial_y b_1 + \partial_x b_2). \end{cases} \quad (4.1)$$

The main difficulty to show the global regularity is H^1-bound for (u,b). We only sketch the proof of H^1 bound.

Taking the inner product of (4.1) with (ω, j) and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_2^2 + \|j\|_2^2\right) + \|\partial_x \omega\|_2^2 + \|\partial_x j\|_2^2 + \epsilon\|\Lambda^\delta \omega\|_2^2 + \epsilon\|\Lambda^\delta j\|_2^2$$
$$=: J_1 + J_2 + J_3 + J_4 + J_5, \quad (4.2)$$

where

$$J_1 = 2\int \partial_x b_1 \partial_y u_1 j \, dx dy, \quad J_2 = 2\int \partial_x b_1 \partial_x u_2 j \, dx dy,$$

$$J_3 = 2\int \partial_x u_1 \partial_y b_1 j \, dx dy, \quad J_4 = 2\int \partial_x u_1 \partial_x b_2 j \, dx dy, \quad J_5 = \int \partial_x \theta \omega \, dx dy.$$

The bounds for J_1, J_2 J_3 and J_4 can be found in [1], which are (for q large enough such that $q\delta > 2$)

$$|J_1| \leq \frac{1}{48}\|\partial_x \omega\|_2^2 + \frac{\epsilon}{4}\|\Lambda^\delta j\|_2^2 + C\|b_1\|_q^{\frac{2q\delta}{q\delta-2}}\|j\|_2^2 + \frac{1}{48}\|\partial_x j\|_2^2$$
$$+ \frac{\epsilon}{4}\|\Lambda^\delta \omega\|_2^2 + C\|b_1\|_q^{\frac{2q\delta}{q\delta-2}}\|\omega\|_2^2,$$

$$|J_2| \leq \frac{1}{48}\|\partial_x \omega\|_2^2 + \frac{1}{48}\|\partial_x j\|_2^2 + C\|\partial_x b_1\|_2^2\left(\|\omega\|_2^2 + \|j\|_2^2\right).$$

We can bound J_3 similar fashion as J_1,

$$|J_3| \leq \frac{1}{48}\|\partial_x\omega\|_2^2 + \frac{1}{48}\|\partial_x j\|_2^2 + \frac{\epsilon}{4}\|\Lambda^\delta j\|_2^2 + C\|u_1\|_q^{\frac{2q\delta}{q\delta-2}}\|j\|_2^2.$$

J_4 can be bounded in a a similar fashion as J_2 and

$$|J_4| \leq \frac{1}{48}\|\partial_x j\|_2^2 + C\|\partial_x u_1\|_2^2\|j\|_2^2.$$

J_5 obeys

$$J_5 \leq \|\theta\|_2\|\partial_x\omega\|_\theta.$$

Inserting the estimates for J_1, J_2, J_3, J_4 and J_5 in (4.2) yields the desired global H^1-bound for (u,b). The H^1- bound for θ is similar to the previous section. One can follow line to line from previous section to show the global H^2-bound. This completes the proof of Theorem 1.3. $\qquad\square$

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