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Global weak solutions for the two-dimensional magnetohydrodynamic equations with partial dissipation and diffusion

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ABSTRACT

In this paper, we consider the two-dimensional magnetohydrodynamic equations. We establish global weak solution for MHD equations with partial dissipation and vertical diffusion. We also obtain a regularity condition.

1. Introduction

The viscous incompressible magnetohydrodynamic equations can be written as

$$\begin{cases} u_t + u \cdot \nabla u = -\nabla p + \nu_1\, \partial_{xx} u + \nu_2\, \partial_{yy} u + b \cdot \nabla b, \\ b_t + u \cdot \nabla b = \eta_1\, \partial_{xx} b + \eta_2\, \partial_{yy} b + b \cdot \nabla u, \\ \nabla \cdot u = 0, \quad \nabla \cdot b = 0, \\ u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y) \end{cases} \tag{1.1}$$

where $(x, y) \in \mathbb{R}^2$, $t \geq 0$, $u = (u_1(x, y, t),\ u_2(x, y, t))$ denotes the 2D velocity field, $p = p(x, y, t)$ the pressure, $b = (b_1(x, y, t), b_2(x, y, t))$ the magnetic field, and ν_1, ν_2, η_1 and η_2 are nonnegative real parameters. When $\nu_1 = \nu_2$ and $\eta_1 = \eta_2$, (1.1) reduces to the standard incompressible MHD equations.

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When all four parameters ν_1, ν_2, η_1 and η_2 are positive, the global regularity of 2D MHD equations is well known [13]. However, it remains a remarkable open problem whether classical solutions of the two-dimensional inviscid MHD equation, all four parameters are zero, preserve their regularity for all time or finite time blowup. When $\nu_1 > 0$, $\nu_2 = 0$, $\eta_1 = 0$ and $\eta_2 > 0$ or when $\nu_1 = 0$, $\nu_2 > 0$, $\eta_1 > 0$ and $\eta_2 = 0$, the global regularity was established by Cao and Wu [2]. Furthermore, the MHD equations with only magnetic diffusion is shown to possess global weak solutions [2]. Cao, Regmi, and Wu studied two dimensional MHD equations with horizontal dissipation and horizontal diffusion in [1]. They proved that any possible blow-up can be controlled by the L^∞ norm of the horizontal components. Furthermore, they showed that $\|v\|_{L^r}$ with $2 < r < \infty$ at any time does not grow faster than $\sqrt{r\log r}$ as r increases i.e. $\|(u_1, b_1)\|_{L^r} \le C\sqrt{r\log r}$. The MHD equations have been a center of attention to numerous analytical, experimental, and numerical investigations (see [1,2,4–20], and references therein).

In this paper, we consider the MHD equations with partial dissipation in the first component of velocity field and vertical magnetic diffusion.

$$\begin{cases} \partial_t u_1 + u\cdot\nabla u_1 = -\partial_x p + \nu_2 \partial_{yy} u_1 + b\cdot\nabla b_1, \\ \partial_t u_2 + u\cdot\nabla u_2 = -\partial_y p + b\cdot\nabla b_2, \\ \partial_t b + u\cdot\nabla b = \eta_1\partial_{xx} b + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y), \qquad b(x,y,0) = b_0(x,y) \end{cases} \tag{1.2}$$

where ν_2 and η_1 are positive parameters, we assume $\nu_2 = \eta_1 = 1$ for simplicity.

The vorticity $\omega = \nabla\times u$ and the current density $j = \nabla\times b$ satisfy

$$\begin{cases} \omega_t + u\cdot\nabla\omega = -\partial_{yyy} u_1 + b\cdot\nabla j, \\ j_t + u\cdot\nabla j = \partial_{xx} j + b\cdot\nabla\omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1). \end{cases} \tag{1.3}$$

We prove the following theorem.

Theorem 1.1. *Assume that $(u_0, b_0) \in H^1(\mathbb{R}^2)$, $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$. Then, (1.2) has a global weak solution (u, b), which satisfies*

$$u, b \in L^\infty([0,T); H^1(\mathbb{R}^2)), \qquad \partial_{yy} u \in L^2([0,T]; L^2(\mathbb{R}^2)), \qquad \partial_x j \in L^2([0,T]; L^2(\mathbb{R}^2))$$

for any $T > 0$.

It is not known if such weak solutions are globally regular. However, if the u_2 satisfies

$$\int_0^T \|\partial_{xx} u_2\|_4 < \infty$$

then the solution becomes a classical solution.

Remark 1.2. • This work is motivated by the recent work in [4,2,1].
• Similar results can be obtained, if we replace $\partial_{yy} u_1$ by $\partial_{xx} u_2$ and $\partial_{xx} b$ by $\partial_{yy} b$ in (1.2).

This paper is organized as follows: we first state some preliminaries and some important lemmas in Section 2 and thereafter in Sections 3 and 4 we prove our main results.

2. Preliminary

Throughout this paper, we use the following notations.

$$\|f\|_{L^2} = \|f\|_2, \qquad \frac{\partial}{\partial x} f = \partial_x f = f_{xx}, \qquad \frac{\partial^2}{\partial x^2} f = \partial_{xx} f, \qquad \int_{\mathbb{R}^2} f\,dxdy = \int f.$$

We state some lemmas which play an important role.

Lemma 2.1 *(See [2]). If $f, g, h, \partial_y g, \partial_x h \in L^2(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} |f\, g\, h|\; dxdy \leq C\, \|f\|_2\, \|g\|_2^{\frac{1}{2}}\, \|\partial_y g\|_2^{\frac{1}{2}}\, \|h\|_2^{\frac{1}{2}}\, \|\partial_x h\|_2^{\frac{1}{2}}, \tag{2.1}$$

where C is a constant.

Lemma 2.2 *(See [3]). Let Ω be divergence-free vector field such that $\nabla\Omega \in L^p$, $p \in (1,\infty)$. Then there exists a constant $C > 0$ such that*

$$\|\nabla\Omega\|_{L^p} \leq C\frac{p^2}{p-1}\|\nabla \times \Omega\|_{L^p}.$$

Lemma 2.3. *The following interpolation inequality holds in $\mathbb{R}^2$.*

$$\|f\|_4 \leq C\|f\|_2^{\frac{1}{2}}\|\nabla f\|_2^{\frac{1}{2}},$$

where C is a constant.

3. Global weak solution

This section establishes the global weak solutions for (1.2).

We can easily prove the global L^2-bound.

Lemma 3.1. *Let $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u, b) be the corresponding solution of (1.2). Then, (u, b) obeys the following global L^2-bound,*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + 2\int_0^t \|\partial_y u_1(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|\partial_x b(\tau)\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \tag{3.1}$$

for any $t \geq 0$.

Proof of Theorem 1.1. Multiplying the first equation in (1.3) by ω and the second equation in (1.3) by j and integrating by parts in $\mathbb{R}^2$, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\omega\|_{L^2}^2 + \|j\|_{L^2}^2\right) + \|\partial_{xy}u_1\|_{L^2}^2 + \|\partial_{yy}u_1\|_{L^2}^2 + \|\partial_x j\|_{L^2}^2\\
&\quad = 2\int \left[\partial_x b_1(\partial_x u_2 + \partial_y u_1)\, j - \partial_x u_1(\partial_x b_2 + \partial_y b_1)\right]\, j dx\, dy.\\
&\quad = J_1 + J_2 + J_3 + J_4\\
J_1 \; &= \int \partial_x b_1 \partial_x u_2 \partial_x b_2 - \int \partial_x b_1 \partial_x u_2 \partial_y b_1\\
&= J_{11} + J_{12}\\
J_{11} \; &\leq \left|\int \partial_x b_1 \partial_x u_2 \partial_x b_2\right|\\
&\leq C\|\partial_x u_2\|\, \|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_x b_2\|_2^{\frac{1}{2}}\|\partial_{xx} b_2\|_2^{\frac{1}{2}}\\
&\leq C\|\omega\|\, \|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_x b_2\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\\
&\leq C\|\omega\|\, \|\partial_x b\|_2 \|\partial_x j\|_2\\
&\leq \frac{1}{48}\|\partial_x j\|_2^2 + C\|\partial_x b\|_2^2\|\omega\|_2^2.
\end{aligned}$$

We have applied Lemma 2.1, Young's inequality, and the simple fact that

$$\|\partial_x b_2\|_{L^2} \leq \|j\|_{L^2}, \qquad \|\partial_{xy} b_1\|_{L^2} \leq \|\partial_x j\|_{L^2},$$

$$\begin{aligned}
J_{12} &= \int \partial_x b_1 \partial_x u_2 \partial_y b_1 = -\int u_2 \partial_{xx} b_1 \partial_y b_1 - \int u_2 \partial_x b_1 \partial_{xy} b_1 \\
&= J_{121} + J_{122} \\
J_{121} &\le \left| -\int u_2 \partial_{xx} b_1 \partial_y b_1 \right| \\
&\le C\|\partial_{xx} b_1\| \, \|u_2\|_2^{\frac{1}{2}} \|\partial_y u_2\|_2^{\frac{1}{2}} \|\partial_y b_1\|_2^{\frac{1}{2}} \|\partial_{xy} b_1\|_2^{\frac{1}{2}} \\
&\le C\|\partial_x j\|_2 \|u_2\|_2^{\frac{1}{2}} \|\omega\|_2^{\frac{1}{2}} \|j\|_2^{\frac{1}{2}} \|\partial_x j\|_2^{\frac{1}{2}} \\
&\le \frac{1}{48}\|\partial_x j\|_2^2 + C\|u_2\| \, \|j\|_2 \|\omega\|_2.
\end{aligned}$$

Similarly,

$$\begin{aligned}
J_{122} &\le \left| -\int u_2 \partial_x b_1 \partial_{xy} b_1 \right| \\
&\le \frac{1}{48}\|\partial_x j\|_2^2 + C\|u_2\| \, \|j\|_2 \|\omega\|_2 \\
J_2 &\le \left| \int \partial_x b_1 \, \partial_y u_1 \, j \right| \\
&\le C\|\partial_x b_1\|_2^{\frac{1}{2}} \|\partial_{xx} b_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_{yy} u_1\|_2^{\frac{1}{2}} \|j\|_2 \\
&\le \frac{1}{48}\left[\|\partial_{yy} u_1\|_2^2 + \|\partial_x j\|_2^2\right] + C(\|\partial_x b_1\|_2^2 + \|\partial_y u_1\|_2^2)\|j\|_2^2 \\
J_3 &\le \left| \int \partial_x u_1 \, \partial_x b_2 \, j \right| \le \int |(u_1 \partial_{xx} b_2 j + u_1 \partial_x b_2 \partial_x j)| \\
&\le C\|u_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|j\|_2^{\frac{1}{2}} \|\partial_x j\|_2^{\frac{1}{2}} \|\partial_{xx} b_2\|_2 + C\|u_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_x b_2\|_2^{\frac{1}{2}} \|\partial_{xx} b_2\|_2^{\frac{1}{2}} \|\partial_x j\|_2 \\
&\le \frac{1}{48}\|\partial_x j\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|j\|_2^2 \\
J_4 &\le \left| \int \partial_x u_1 \, \partial_y b_1 \, j \right| \le \int |(u_1 \partial_{xy} b_1 j - u_1 \partial_y b_1 j_x)| \\
&\le C\|u_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|j\|_2^{\frac{1}{2}} \|\partial_x j\|_2^{\frac{1}{2}} \|\partial_{xy} b_1\|_2 + C\|u_1\|_2^{\frac{1}{2}} \|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_y b_1\|_2^{\frac{1}{2}} \|\partial_{xy} b_1\|_2^{\frac{1}{2}} \|j_x\|_2 \\
&\le \frac{1}{48}\|\partial_x j\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|j\|_2^2.
\end{aligned}$$

Combining these estimates, and using Gronwall's inequality, we have

$$\|\omega\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \|j\|_{L^\infty(0,T;L^2(\mathbb{R}^2))} + \|\nabla \partial_y u_1\|_{L^2(0,T;L^2(\mathbb{R}^2))} + \|\partial_x j\|_{L^2(0,T;L^2(\mathbb{R}^2))} \le C.$$

This completes the proof of Theorem 1.1.

4. Global regularity of weak solution

This section focuses on the global regularity of weak solution. More precisely, we prove the following theorem.

Theorem 4.1. *Assume that* $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ *and* $\nabla \cdot b_0 = 0$. *If, for some* $T > 0$, $\int_0^T \|\partial_{xx} u_2\|_4 < \infty$. *Then,* (1.2) *has a unique global solution* (u, b) *satisfying* $(u, b) \in L^\infty([0, T]; H^2(\mathbb{R}^2))$.

Proof. The local well-posedness follows from a standard approach, the proof of Theorem 4.1 reduces to obtaining a global *a priori* bound for $\|(u, b)\|_{H^2}$.

Taking the inner products of (1.3) with $\Delta\omega$ and Δj, adding and integrating by parts, we obtain.

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2) + \|\Delta \partial_y u_1\|_2^2 + \|\nabla \partial_x j\|_2^2 = L_1 + L_2 + L_3 + L_4 + L_5 \tag{4.1}$$

where

$$L_1 = -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \, dxdy, \qquad L_2 = -\int \nabla j \cdot \nabla u \cdot \nabla j \, dxdy,$$
$$L_3 = 2\int \nabla\omega \cdot \nabla b \cdot \nabla j \, dxdy, \qquad L_4 = 2\int \nabla[\partial_x b_1(\partial_x u_2 + \partial_y u_1)] \cdot \nabla j \, dxdy,$$
$$L_5 = -2\int \nabla[\partial_x u_1(\partial_x b_2 + \partial_y b_1)] \cdot \nabla j \, dxdy.$$

$$\begin{aligned} L_1 &= -\int (\nabla\omega \cdot \nabla u \cdot \nabla\omega) \, dxdy \\ &= -\int \left(\partial_x u_1(\partial_x\omega)^2 + \partial_x u_2 \partial_x\omega\partial_y\omega + \partial_y u_1\partial_x\omega\partial_y\omega + \partial_y u_2(\partial_y\omega)^2\right) \\ &= L_{11} + L_{12} + L_{13} + L_{14} \end{aligned}$$

$$L_{11} = -\int \partial_x u_1(\partial_{xx}u_2)^2 - \int \partial_x u_1(\partial_{xy}u_1)^2 + 2\int \partial_x u_1 \partial_{xx}u_2\partial_{xy}u_1$$

$$\begin{aligned} \left|\int \partial_x u_1(\partial_{xx}u_2)^2\right| &\le C\|\partial_{xx}u_2\|_4\|\partial_{xx}u_2\|_2\|\partial_x u_1\|_4 \\ &\le C\|\partial_{xx}u_2\|_4\|\partial_{xx}u_2\|_2\|\partial_x u_1\|_2^{\frac{1}{2}}\|\nabla\partial_x u_1\|_2^{\frac{1}{2}} \\ &\le C\|\partial_{xx}u_2\|_4\|\nabla\omega\|_2^{\frac{3}{2}}\|\partial_x u_1\|_2^{\frac{1}{2}} \\ &\le C\|\partial_{xx}u_2\|_4(\|\nabla\omega\|_2^2 + \|\omega\|_2^2). \end{aligned}$$

We have applied Young's inequality, Theorem 4.1.

$$\begin{aligned} \left|\int \partial_x u_1(\partial_{xy}u_1)^2\right| &= \left|-2\int u_1\partial_{xy}u_1\partial_{xxy}u_1\right| \\ &\le C\|\partial_{xxy}u_1\|_2\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{xxy}u_1\|_2^{\frac{1}{2}} \\ &\le C\|\partial_{xyy}u_2\|_2\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\Delta\partial_y u_1\|_2^{\frac{1}{2}} \\ &\le C\|\partial_{yy}\omega\|_2^{\frac{3}{2}}\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}} \\ &\le \frac{1}{48}\|\partial_{yy}\omega\|_2^2 + \|u_1\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2 \end{aligned}$$

$$\begin{aligned} \left|2\int \partial_x u_1\partial_{xx}u_2\partial_{xy}u_1\right| &\le C\|\partial_x u_1\|_2\|\partial_{xx}u_2\|_4\|\partial_{xy}u_2\|_4 \\ &\le C\|\partial_{xx}u_2\|_4(\|\nabla\omega\|_2^2 + \|\omega\|_2^2) \end{aligned}$$

$$\begin{aligned} L_{12} &= \int \partial_x u_2\partial_x\omega\partial_y\omega \\ &= \int \partial_x u_2\partial_{xx}u_2\partial_{xy}u_2 - \int \partial_x u_2\partial_{xx}u_2\partial_{yy}u_1 - \int \partial_x u_2\partial_{xy}u_1\partial_{xy}u_2 - \int \partial_x u_2\partial_{xy}u_1\partial_{yy}u_1 \end{aligned}$$

$$\begin{aligned} \left|\int \partial_x u_2\partial_{xx}u_2\partial_{xy}u_2 - \int \partial_x u_2\partial_{xx}u_2\partial_{yy}u_1\right| &\le C\|\partial_{xx}u_2\|_4\|\partial_x u_2\|_4\|\partial_{xy}u_2\|_2 + C\|\partial_{xx}u_2\|_4\|\partial_x u_2\|_4\|\partial_{yy}u_1\|_2 \\ &\le C\|\partial_{xx}u_2\|_4(\|\nabla\omega\|_2^2 + \|\omega\|_2^2) \end{aligned}$$

$$\begin{aligned} \left|\int \partial_x u_2\partial_{xy}u_1\partial_{xy}u_2 - \int \partial_x u_2\partial_{xy}u_1\partial_{yy}u_1\right| &\le C\|\partial_{xy}u_1\|_2\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xx}u_2\|_2^{\frac{1}{2}}\|\partial_{xy}u_2\|_2^{\frac{1}{2}}\|\partial_{xyy}u_2\|_2^{\frac{1}{2}} \\ &\quad + C\|\partial_{xy}u_1\|_2\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xy}u_2\|_2^{\frac{1}{2}}\|\partial_{yy}u_1\|_2^{\frac{1}{2}}\|\partial_{yyy}u_1\|_2^{\frac{1}{2}} \\ &\le C\|\partial_{xy}u_1\|_2\|\partial_x u_2\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{1}{2}}\|\partial_{yy}\omega\|_2^{\frac{1}{2}} \\ &\le \frac{1}{48}\|\partial_{yy}\omega\|_2^2 + (\|\partial_x u_2\|_2^2 + \|\partial_{xy}u_1\|_2^2 + 1)\|\nabla\omega\|_2^2 \end{aligned}$$

$$\begin{aligned}
L_{13} &\le \left|\int \partial_y u_1 \partial_x \omega \partial_y \omega\right| \\
&\le C\|\partial_y u_1\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\partial_x \omega\|_2\|\partial_y \omega\|_2^{\frac12}\|\partial_{yy}\omega\|_2^{\frac12} \\
&\le C\|\partial_{yy}\omega\|_2\|\partial_{xy}u_1\|_2\|\partial_y\omega\|_2 + C\|\nabla\omega\|_2^2 \\
&\le \frac{1}{48}\|\partial_{yy}\omega\|_2^2 + C(\|\partial_{xy}u_1\|_2^2+1)\|\nabla\omega\|_2^2 \\
L_{14} &\le \left|2\int u_2 \partial_y\omega\partial_{yy}\omega\right| \\
&\le C\|\partial_{yy}\omega\|_2\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\partial_y\omega\|_2^{\frac12}\|\partial_{yy}\omega\|_2^{\frac12} \\
&\le C\|\partial_{yy}\omega\|_2^{\frac32}\|u_2\|_2^{\frac12}\|\omega\|_2^{\frac12}\|\nabla\omega\|_2^{\frac12} \\
&\le \frac{1}{48}\|\partial_{yy}\omega\|_2^2 + C\|u_2\|_2^2\|\omega\|_2^2\|\nabla\omega\|_2^2 \\
L_2 &= -\int \nabla j\cdot\nabla u\cdot\nabla j\, dxdy \\
&= -\int \big(\partial_x u_1(\partial_x j)^2 + \partial_y u_1\partial_x j\partial_y j + \partial_y u_2(\partial_y j)^2 + \partial_x u_2\partial_x j\partial_y j\big) \\
&= L_{21}+L_{22}+L_{23}+L_{24} \\
L_{21} &\le C\|\partial_x j\|_2\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xx}j\|_2^{\frac12} \\
&\le C\|\omega\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\nabla j\|^{\frac32}\|\nabla\partial_x j\|_2^{\frac12} \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\partial_{xy}u_1\|_2^{\frac23}\|\partial_x\omega\|_2^{\frac23}\|\nabla j\|_2^2 \\
L_{22} &\le C\|\partial_y u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}\|\partial_y j\|_2 \\
&\le C\|\omega\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\nabla j\|_2^{\frac32}\|\nabla\partial_x j\|_2^{\frac12} \\
L_{23} &\le C\|\partial_y j\|_2\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_y j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12} \\
&\le C\|\nabla j\|_2^{\frac32}\|\nabla\partial_x j\|_2^{\frac12}\|\omega\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12} \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + \|\omega\|_2^{\frac23}\|\partial_{xy}u_1\|_2^{\frac23}\|\nabla j\|_2^2 \\
L_{24} &\le C\|\partial_x u_2\|_2\|\partial_y j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12} \\
&\le C\|\omega\|_2\|\nabla j\|_2\|\nabla\partial_x j\|_2 \\
L_3 &= \int \partial_x\omega\partial_x b_1\partial_x j + \partial_x\omega\partial_x b_2\partial_y j + \partial_y\omega\partial_y b_1 j_x + \partial_y\omega\partial_y b_2\partial_y j \\
&= L_{31}+L_{32}+L_{33}+L_{34} \\
L_{31} &\le \left|\int \partial_x\omega\partial_x b_1\partial_x j\right| \\
&\le C\|\partial_x\omega\|_2\|\partial_x b_1\|_2^{\frac12}\|\partial_{xx}b_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12} \\
&\le C\|\partial_x\omega\|_2\|\partial_x b_1\|_2^{\frac12}\|\nabla j\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12} \\
&\le C\|\partial_x j\|_2\|\nabla\partial_x j\|_2 + \|\partial_x j\|\,\|\partial_x b_1\|\,\|\partial_x\omega\|_2^2 \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C(\|\partial_x b_1\|_2^2 + \|\partial_x j\|_2^2+1)(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2) \\
L_{32} &\le \frac{1}{48}\|\nabla j_x\|_2^2 + C(\|\partial_x b_2\|_2^2 + \|\partial_x j\|^2+1)(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2) \\
L_{33} &\le C\|\partial_y b_1\|_2^{\frac12}\|\partial_{xy}b_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}\|\nabla\omega\|_2 \\
&\le C\|j\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\nabla j\|_2^{\frac12}\|\nabla\partial_x j\|_2^{\frac12}\|\nabla\omega\|_2 \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\partial_x j\|_2^2\|\nabla j\|_2^2 + C\|j\|_2\|\nabla\omega\|_2^2
\end{aligned}$$

$$
\begin{aligned}
L_{34} &\le C\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}}\|\nabla\omega\|_2 \\
&\le C\|j\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2^{\frac{1}{2}}\|\nabla\omega\|_2 \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\partial_x j\|_2^2\|\nabla j\|_2^2 + C\|j\|_2\|\nabla\omega\|_2^2 \\
L_4 &= 2\int \nabla[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]\cdot\nabla j\; dxdy \\
&= 2\int \partial_x[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]j_x + \partial_y[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]j_y\; dxdy \\
&= L_{41} + L_{42} \\
L_{41} &\le \left|-2\int \partial_x b_1(\partial_x u_2 + \partial_y u_1)\partial_{xx} j\right| \\
&\le C(\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xy} u_2\|_2^{\frac{1}{2}} + C\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy} u_1\|_2^{\frac{1}{2}})\|\partial_{xx} j\|_2 \\
&\le C\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\omega\|_2^{\frac{1}{2}}\|\omega_y\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2 \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\omega\|_2\|j\|_2(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2) \\
L_{42} &= 2\int (\partial_{xy} b_1\partial_x u_2 + \partial_x b_1\partial_{xy} u_2 + \partial_{xy} b_1\partial_y u_1 + \partial_x b_1\partial_{yy} u_1)\partial_y j\; dxdy \\
&= L_{421} + L_{422} + L_{423} + L_{424} \\
L_{421} &\le C\|\partial_x u_2\|_2\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_{xyy} b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\
&\le C\|\omega\|_2\|\nabla j\|_2\|\nabla\partial_x j\|_2 \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\omega\|_2^2\|\nabla j\|_2^2 \\
L_{422} &\le C\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_{xy} u_2\|_2\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\
&\le C\|j\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_x\omega\|\,\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2^{\frac{1}{2}} \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\partial_x j\|_2^2\|\nabla j\|_2^2 + C\|j\|_2\|\nabla\omega\|_2^2 \\
L_{423} &\le C\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_{xyy} b_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy} u_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2 \\
&\le C\|\partial_x j\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2^{\frac{1}{2}}\|\omega\|_2^{\frac{1}{2}}\|\partial_x\omega\|_2^{\frac{1}{2}}\|\nabla j\|_2 \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\omega\|_2\|\nabla j\|_2^2 + C\|\partial_x j\|_2^2\|\nabla\omega\|_2^2 \\
L_{424} &\le C\|\partial_y j\|_2\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_{yy} u_1\|_2^{\frac{1}{2}}\|\partial_{xyy} u_1\|_2^{\frac{1}{2}} \\
&\le C\|\nabla j\|_2\|j\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{1}{2}}\|\partial_{yy}\omega\|_2^{\frac{1}{2}} \\
&\le \frac{1}{48}\|\partial_{yy}\omega\|_2^2 + C\|\partial_x j\|_2^2\|\nabla\omega\|_2^2 + C\|j\|_2\|\nabla j\|_2^2 \\
L_5 &= -2\int \nabla[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\cdot\nabla j\; dxdy \\
&= -2\int \partial_x[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\partial_x j + \partial_y[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\partial_y j\; dxdy \\
&= L_{51} + L_{52} \\
L_{51} &\le C\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy} u_1\|_2^{\frac{1}{2}}\|\partial_x b_2\|_2^{\frac{1}{2}}\|\partial_{xx} b_2\|_2^{\frac{1}{2}}\|\partial_{xx} j\|_2 + C\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy} u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_{xx} j\|_2^{\frac{1}{2}} \\
&\le C\|\omega\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2 \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\omega\|_2\|j\|_2(\|\nabla\omega\|_2^2 + \|\nabla j\|_2^2) \\
L_{52} &= -2\int (\partial_{xy} u_1\partial_x b_2 + \partial_x u_1\partial_{xy} b_2 + \partial_{xy} u_1\partial_y b_1 + \partial_x u_1\partial_{yy} b_1)\partial_y j\; dxdy \\
&= L_{521} + L_{522} + L_{523} + L_{524}
\end{aligned}
$$

L_{521} and L_{522} can be bounded as,

$$
\begin{aligned}
L_{521} &\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C(\|\partial_x j\|_2^2 + \|j\|_2^2)(\|\nabla j\|_2^2 + \|\nabla\omega\|_2^2)\\
L_{522} &\le \frac{1}{48}\|\nabla\partial_x\|_2^2 + C\|\omega\|_2^{\frac{2}{3}}\|\partial_{xy}u_1\|_2^{\frac{2}{3}}\|\nabla j\|_2^2\\
L_{523} &\le C\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{xyy}u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy}b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2\\
&\le C\|\omega_y\|_2^{\frac{1}{2}}\|\omega_{yy}\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|j_x\|_2^{\frac{1}{2}}\|\nabla j\|_2\\
&\le C\|\omega_{yy}\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\nabla\omega\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2\\
&\le \frac{1}{48}\|\partial_{yy}\omega\|_2^2 + C\|\partial_x j\|_2^2\|\nabla\omega\|_2^2 + C\|j\|_2\|\nabla j\|_2^2\\
L_{524} &\le C\|\partial_y j\|_2\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{yy}b_1\|_2^{\frac{1}{2}}\|\partial_{xyy}b_1\|_2^{\frac{1}{2}}\\
&\le C\|\nabla j\|_2\|\omega\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2^{\frac{1}{2}}\\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\partial_{xy}u_1\|_2^2\|\nabla j\|_2^2 + \|\omega\|_2\|\nabla j\|_2^2.
\end{aligned}
$$

Inserting the estimates for $L_1, L_2, L_3, L_4 \& L_5$ in (4.1) and applying Gronwall's inequality, yields the desired bound. This completes the proof of Theorem 4.1. □

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