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The 2D magneto-micropolar equations with partial dissipation

Dipendra Regmi

Department of Mathematics, University of North Georgia, Oakwood, Georgia

Correspondence
Dipendra Regmi, Department of Mathematics, University of North Georgia, 3820 Mundy Mill Rd, Oakwood, GA 30566.
Email: dipendra.regmi@ung.edu

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We study the global existence and regularity of classical solutions to the 2D incompressible magneto-micropolar equations with partial dissipation. We establish the global regularity for one partial dissipation case. The proofs of our main results rely on anisotropic Sobolev type inequalities and suitable combination and cancellation of terms.

KEYWORDS
global regularity, magneto-micropolar equations, partial dissipation

1 | INTRODUCTION

The purpose of this paper is to study the global existence and regularity of classical solutions to the 2D incompressible magneto-micropolar equations with partial dissipation. The standard 3D incompressible magneto-micropolar equations can be written as

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla(p + \frac{1}{2}|b|^2) = (\mu+\chi)\Delta u + (b\cdot\nabla)b + 2\chi\nabla\times\omega, \\ \partial_t b + (u\cdot\nabla)b = \nu\Delta b + (b\cdot\nabla)u, \\ \partial_t\omega + (u\cdot\nabla)\omega + 2\chi\omega = \kappa\Delta\omega + (\alpha+\beta)\nabla\nabla\cdot\omega + 2\chi\nabla\times u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \end{cases} \tag{1}$$

where, for $\mathbf{x}\in\mathbb{R}^3$ and $t\geq 0$, $u = u(\mathbf{x},t), b = b(\mathbf{x},t), \omega = \omega(\mathbf{x},t)$ and $p = p(\mathbf{x},t)$ denote the velocity field, the magnetic field, the microrotation field, and the pressure respectively, and μ denotes the kinematic viscosity, ν the magnetic diffusivity, χ the vortex viscosity, and α, β, and κ the angular viscosities.

The 3D magneto-micropolar equations reduce to the 2D magneto-micropolar equations when

$$u = (u_1(x,y,t), u_2(x,y,t), 0), \quad b = (b_1(x,y,t), b_2(x,y,t), 0),$$
$$\omega = (0,0,\omega(x,y,t)), \quad \pi = \pi(x,y,t),$$

where $(x,y)\in\mathbb{R}^2$, and we have written $\pi = p + \frac{1}{2}|b|^2$. The 2D magneto-micropolar equations can be written as

$$\begin{cases} \partial_t u + (u\cdot\nabla)u + \nabla\pi = (\mu+\chi)\Delta u + (b\cdot\nabla)b + 2\chi\nabla\times\omega, \\ \partial_t b + (u\cdot\nabla)b = \nu\Delta b + (b\cdot\nabla)u, \\ \partial_t\omega + (u\cdot\nabla)\omega + 2\chi\omega = \kappa\Delta\omega + 2\chi\nabla\times u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \end{cases} \tag{2}$$

where $u = (u_1,u_2), b = (b_1,b_2), \nabla\times\omega = (-\partial_y\omega, \partial_x\omega)$ and $\nabla\times u = \partial_x u_2 - \partial_y u_1$.

A generalization of the 2D magneto-micropolar equations can be written as

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x\pi = \mu_{11}\partial_{xx}u_1 + \mu_{12}\partial_{yy}u_1 + (b\cdot\nabla)b_1 - 2\chi\partial_y\omega, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y\pi = \mu_{21}\partial_{xx}u_2 + \mu_{22}\partial_{yy}u_2 + (b\cdot\nabla)b_2 + 2\chi\partial_x\omega, \\ \partial_t b_1 + (u\cdot\nabla)b_1 = \nu_{11}\partial_{xx}b_1 + \nu_{12}\partial_{yy}b_1 + (b\cdot\nabla)u_1, \\ \partial_t b_2 + (u\cdot\nabla)b_2 = \nu_{21}\partial_{xx}b_2 + \nu_{22}\partial_{yy}b_2 + (b\cdot\nabla)u_2, \\ \partial_t\omega + (u\cdot\nabla)\omega + 2\chi\omega = \kappa_1\partial_{xx}\omega + \kappa_2\partial_{yy}\omega + 2\chi\nabla\times u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y),\ \omega(x,y,0) = \omega_0(x,y). \end{cases} \tag{3}$$

If $\mu_{11} = \mu_{12} = \mu_{21} = \mu_{21} = \mu, \nu_{11} = \nu_{12} = \nu_{21} = \nu_{21} = \nu$, and $\kappa_1 = \kappa_2 = \kappa$, then (3) reduces to the standard magneto-micropolar equations (2). For notational convenience, we set $\chi = \frac{1}{2}$ for the rest of the paper.

The magneto-micropolar equations model the motion of electrically conducting micropolar fluids in the presence of a magnetic field. The above generalization is capable of modeling the motion of anisotropic fluids for which the diffusion properties in different directions are different. Furthermore, (3) allows us to explore the smoothing effects of various partial dissipations.

The equations for $\Omega = \nabla\times u$, the current density $j = \nabla\times b$, and $\nabla\omega$ can be expressed as

$$\begin{cases} \Omega_t + u\cdot\nabla\Omega = -\mu_{11}\partial_{xxy}u_1 - \mu_{12}\partial_{yyy}u_1 + \mu_{21}\partial_{xxx}u_2 + \mu_{22}\partial_{xyy}u_2 + (b\cdot\nabla)j - \Delta\omega, \\ j_t + u\cdot\nabla j = -\nu_{11}\partial_{xxy}b_1 - \nu_{12}\partial_{yyy}b_1 + \nu_{21}\partial_{xxx}b_2 + \nu_{22}\partial_{xyy}b_2 + b\cdot\nabla\Omega \\ \qquad +2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \\ \partial_t\nabla\omega + \nabla(u\cdot\nabla\omega) + \nabla\omega = \kappa_1\nabla\omega_{xx} + \kappa_2\nabla\omega_{yy} + \nabla\Omega, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0. \end{cases} \tag{4}$$

The global regularity has been established for the following three cases by D. Regmi and J. Wu in their study.[1]

- $\mu_{11} = \mu_{22} = 0, \nu_{21} = \nu_{22} = 0, \kappa_2 = 0, \mu_{12} = \mu_{21} = 1, \kappa_1 = \nu_{11} = \nu_{12} = 1,$
- $\mu_{11} = \mu_{12} = 1, \nu_{21} = \nu_{22} = 0, \kappa_2 = 0, \mu_{22} = \mu_{21} = 0, \kappa_1 = \nu_{11} = \nu_{12} = 1,$
- $\mu_{12} = \mu_{22} = 1, \nu_{21} = \nu_{22} = 0, \kappa_2 = 0, \mu_{11} = \mu_{21} = 0, \kappa_1 = \nu_{11} = \nu_{12} = 1.$

In addition, the global regularity of the following two cases has been settled recently by Y. Guo and H. Shang in their study.[2]

- $\mu_{11} = \mu_{22} = 0, \nu_{21} = \nu_{22} = 0, \kappa_1 = 0, \mu_{12} = \mu_{21} = 1, \kappa_2 = \nu_{11} = \nu_{12} = 1,$
- $\mu_{11} = \mu_{21} = 1, \nu_{12} = \nu_{21} = 1, \kappa_2 = 1, \mu_{12} = \mu_{22} = 0, \kappa_1 = \nu_{11} = \nu_{22} = 0.$

In this paper, we study the regularity of the following three cases.

Case 1: $\mu_{11} = \mu_{12} = 0, \nu_{11} = \nu_{21} = 0, \kappa_1 = 0, \mu_{21} = \mu_{22} = 1, \kappa_2 = \nu_{12} = \nu_{22} = 1.$
More precisely,

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x\pi = (b\cdot\nabla)b_1 - \partial_y\omega, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y\pi = \partial_{xx}u_2 + \partial_{yy}u_2 + (b\cdot\nabla)b_2 + \partial_x\omega, \\ \partial_t b_1 + (u\cdot\nabla)b_1 = \partial_{yy}b_1 + (b\cdot\nabla)u_1, \\ \partial_t b_2 + (u\cdot\nabla)b_2 = \partial_{yy}b_2 + (b\cdot\nabla)u_2, \\ \partial_t\omega + (u\cdot\nabla)\omega + \omega = \partial_{yy}\omega + \nabla\times u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y),\ \omega(x,y,0) = \omega_0(x,y). \end{cases} \tag{5}$$

Case 2: $\mu_{11} = \mu_{12} = \mu_{22} = 0, \nu_{11} = \nu_{21} = 0, \kappa_2 = 0, \mu_{21} = 1, \kappa_1 = \nu_{12} = \nu_{22} = 1.$
More precisely,

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x\pi = (b\cdot\nabla)b_1 - \partial_y\omega, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y\pi = \partial_{xx}u_2 + (b\cdot\nabla)b_2 + \partial_x\omega, \\ \partial_t b_1 + (u\cdot\nabla)b_1 = \partial_{yy}b_1 + (b\cdot\nabla)u_1, \\ \partial_t b_2 + (u\cdot\nabla)b_2 = \partial_{yy}b_2 + (b\cdot\nabla)u_2, \\ \partial_t\omega + (u\cdot\nabla)\omega + \omega = \partial_{xx}\omega + \nabla\times u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y),\ \omega(x,y,0) = \omega_0(x,y). \end{cases} \tag{6}$$

Case 3: $\mu_{11} = \mu_{21} = \mu_{22} = 0, \nu_{11} = \nu_{21} = 0, \kappa_1 = 0, \mu_{12} = 1, \kappa_2 = \nu_{12} = \nu_{22} = 1.$

More precisely,

$$\begin{cases} \partial_t u_1 + (u\cdot\nabla)u_1 + \partial_x \pi = \partial_{yy} u_1 + (b\cdot\nabla)b_1 - \partial_y \omega, \\ \partial_t u_2 + (u\cdot\nabla)u_2 + \partial_y \pi = (b\cdot\nabla)b_2 + \partial_x \omega, \\ \partial_t b_1 + (u\cdot\nabla)b_1 = \partial_{yy} b_1 + (b\cdot\nabla)u_1, \\ \partial_t b_2 + (u\cdot\nabla)b_2 = \partial_{yy} b_2 + (b\cdot\nabla)u_2, \\ \partial_t \omega + (u\cdot\nabla)\omega + \omega = \partial_{yy}\omega + \nabla\times u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y),\, b(x,y,0) = b_0(x,y),\, \omega(x,y,0) = \omega_0(x,y). \end{cases} \tag{7}$$

In this paper, we establish the global regularity for the (5). We prove the following theorems.

Theorem 1.1. *Assume $(u_0, b_0, \omega_0) \in H^2(\mathbb{R}^2)$, and $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Then, (5) has a unique global classical solution (u, b, ω) satisfying, for any $T > 0$,*

$$(u, b, \omega) \in L^\infty([0,T]; H^2(\mathbb{R}^2)).$$

The global regularity is extremely hard in the cases 2 and 3. The dissipative term is not sufficient to control nonlinear term in (6) and (7).

We prove the following result.

Proposition 1.2. *Assume $(u_0, b_0, \omega_0) \in H^1(\mathbb{R}^2)$, and $\Delta\cdot u_0 = \Delta\cdot b_0 = 0$. Then, (6) has a unique global classical solution (u, b, ω) satisfying, for any $T > 0$,*

$$(u, b, \omega) \in L^\infty([0,T]; H^1(\mathbb{R}^2))$$

provided $\int_0^T \|\partial_y u\|_{BMO}\, dt < \infty$.

One can easily prove the following result.

Proposition 1.3. *Assume $(u_0, b_0, \omega_0) \in H^1(\mathbb{R}^2)$, and $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Then, (7) has a unique global classical solution (u, b, ω) satisfying, for any $T > 0$,*

$$(u, b, \omega) \in L^\infty([0,T]; H^1(\mathbb{R}^2))$$

provided $\int_0^T \|\partial_x u\|_{BMO}\, dt < \infty$.

When $\omega \equiv 0$, the magneto-micropolar equations become the magneto-hydrodynamic (MHD) equations. When $b \equiv 0$, the magneto-micropolar equations become the micropolar equations. It is worth mentioning that all the results presented here are true for aforementioned equations. There are several global regularity results for the 2D MHD, and the 2D micropolar equations with partial dissipation are available (see, eg, the previous studies[3-22]).

The mathematical study of the magneto-micropolar equations started in the seventies and has been continued by many authors (see, eg, the previous studies[2,23-30]). Some of the recent efforts are devoted to the well-posedness problem and various asymptotic behavior. In his study,[26] Yamazaki obtained the global regularity of the 2D magneto-micropolar equation with zero angular viscosity, namely, (2) with $\kappa = 0$ and other coefficients being positive. Another partial dissipation case for the 2D magneto-micropolar equation was studied in Cheng and Liu.[31]

The main idea to establish the global existence and regularity results consists of two steps. First step is to show local well-possedness, and the second step is extending the local solution into a global one by obtaining global (in time) a priori bounds. The local well-posedness follows from a classical approach, and we omit here. The main difficulty is global a priori bounds. Thus, we mainly concentrate on the global bounds. The main difficulty is obtaining H^1 bound because of partial dissipation. The rest of this paper is divided into three sections. The last two sections are devoted to the proof of theorem 1.1 and proposition 1.2.

2 | PRELIMINARIES

To simplify the notation, we will write $\|f\|_2$ for $\|f\|_{L^2}$, $\int f$ for $\int_{\mathbb{R}^2} f\, dxdy$ and write $\frac{\partial}{\partial x} f$, $\partial_x f$ or f_x as the first partial derivative and $\frac{\partial^2 f}{\partial x^2}$ or $\partial_{xx} f$ as the second partial throughout the rest of this paper. BMO represents the bounded mean oscillation.

The following anisotropic type Sobolev inequality will be frequently used. Its proof can be found in Cao and Wu.[4]

Lemma 1. *If $f, g, h, \partial_y g, \partial_x h \in L^2(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} |fgh| \, dxdy \le C \|f\|_2 \|g\|_2^{\frac{1}{2}} \|\partial_y g\|_2^{\frac{1}{2}} \|h\|_2^{\frac{1}{2}} \|\partial_x h\|_2^{\frac{1}{2}}, \tag{8}$$

where C is a constant.

The following simple fact on the boundedness of Riesz transforms will also be used. Its proof can be found in Chemin.[32]

Lemma 2. *Let f be divergence-free vector field such that $\Delta f \in L^p$ for $p \in (1, \infty)$. Then, there exists a pure constant $C > 0$ (independent of p) such that*

$$\|\nabla f\|_{L^p} \le \frac{Cp^2}{p-1} \|\nabla \times f\|_{L^p}.$$

3 | PROOF OF THEOREM 1.2

We first prove the global L^2-bound.

Lemma 3.1. *Assume that (u_0, b_0, ω_0) satisfies the condition stated in Theorem 1.1. Let (u, b, ω) be the corresponding solution of (5). Then, for any $T > 0$, (u, b, ω) obeys the following global L^2-bound:*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + 2\int_0^T \|\partial_x u_2, \partial_y u_2\|_{L^2}^2 d\tau$$
$$+ 2\int_0^T \|\partial_y b(\tau)\|_{L^2}^2 d\tau + 2\int_0^T \|\partial_y \omega(\tau)\|_{L^2}^2 d\tau \le C(\|(u_0, b_0, \omega_0)\|_2^2),$$

where C depends on T and initial data.

Proof. Taking the L^2-inner product of (u, b, ω) with (5), respectively, yields

$$\frac{1}{2}\frac{d}{dt}(\|u(t), \omega(t), b(t)\|_2^2) + \|\partial_x u_2(\tau), \partial_y u_2(\tau)\|_2^2 + \|\partial_y b_1(\tau), \partial_y b_2(\tau)\|_2^2$$
$$+ \|\omega_y(\tau)\|_2^2 + \|\omega(\tau)\|_2^2 = 2\left[\int (\partial_x u_2 - \partial_y u_1)\omega \, dx\, dy\right]. \tag{9}$$

The right hand side can be bounded as

$$2\left[\int (\partial_x u_2 - \partial_y u_1)\omega \, dx\, dy\right] \le \frac{1}{2}\|\partial_x u_2\|_2^2 + \frac{1}{2}\|\omega_y\|_2^2 + C(\|\omega\|_2^2 + \|u\|_2^2).$$

After applying Gronwall's inequality, we obtain

$$\frac{1}{2}(\|u(t), \omega(t), b(t)\|_2^2) + \int_0^T \|\partial_x u_2(\tau), \partial_y u_2(\tau)\|_2^2) \, d\tau + \int_0^T (\|\partial_y b_1(\tau), \partial_y b_2(\tau)\|_2^2) \, d\tau$$
$$+ \int_0^T (\|\omega_y(\tau)\|_2^2 + \|\omega(\tau)\|_2^2) \, d\tau \le C.$$

□

The next step is to prove the global H^1-bound for u, b, and ω.

Proposition 3.2. *Assume that (u_0, b_0, ω_0) satisfies the condition stated in Theorem 1.1. Let (u, b, ω) be the corresponding solution of (5). Then, (u, b, ω) satisfies, for any $T > 0$,*

$$(u, b, \omega) \in C([0, T]; H^1). \tag{10}$$

Proof. To estimate the H^1-norm of (u, b, ω), we consider the equations of $\Omega = \nabla \times u$, $\nabla \omega$ and of the current density $j = \nabla \times b$,

$$\Omega_t + u \cdot \nabla \Omega = \partial_{xxx} u_2 + \partial_{xyy} u_2 + (b \cdot \nabla) j - \Delta \omega, \tag{11}$$

$$j_t + u \cdot \nabla j = -\partial_{yyy} b_1 + \partial_{xyy} b_2 + b \cdot \nabla \Omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \tag{12}$$

$$\partial_t \nabla\omega + \nabla(u\cdot\nabla\omega) + \nabla\omega = \nabla\omega_{yy} + \nabla\Omega, \qquad (13)$$
$$\nabla\cdot u = 0, \quad \nabla\cdot b = 0.$$

Dotting (11) by Ω, (12) by j, and (13) by $\Delta\omega$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \|\nabla\omega\|_2^2\right) + \|\nabla\partial_x u_2, \nabla\partial_y u_2\|_2^2 + \|\partial_y j\|_{L^2}^2 + \|\nabla\omega_y\|_2^2 + \|\nabla\omega\|_2^2$$
$$= 2\int\left[\partial_x b_1(\partial_x u_2 + \partial_y u_1) j - \partial_x u_1(\partial_x b_2 + \partial_y b_1)\right] j dx dy - \int \nabla\omega\cdot\nabla u\cdot\nabla\omega + \int \nabla\omega\cdot\nabla\Omega$$
$$\equiv J_1 + J_2 + J_3 + J_4 + J_5 + J_6.$$

We now estimate the terms on the right.

$$J_1 = \int \partial_x b_1 \partial_x u_2 j \le C\|\partial_x b_1\|_2^{\frac12}\|\partial_{xx} b_1\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\partial_{xy} u_2\|_2^{\frac12}\|j\|_2$$
$$\le \frac{1}{48}(\|\nabla\partial_y b\|_2^2 + \|\nabla\partial_y u_2\|_2^2) + C(\|\partial_x u_2\|_2^2 + \|\partial_y b_2\|_2^2)\|j\|_2^2.$$

Similarly,

$$J_2 \le \left|\int \partial_x b_1\, \partial_y u_1\, j\right| = \left|\int \partial_x b_1\, \partial_y u_1 \quad (\partial_x b_2 - \partial_y b_1)\right| \le \frac{1}{48}\|\partial_y j\|_2^2 + C(\|u_1\|_2^2\|\partial_y b\|_2^2 + 1)\|\Omega\|_2^2.$$

J_2, J_3, and J_4 can be bounded by

$$J_3 \le \left|\int \partial_x u_1\, \partial_x b_2\, j\right| \le \int\left|(u_2\partial_{xy} b_2 j + u_2\partial_x b_2 \partial_y j)\right| \le C\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|j\|_2^{\frac12}\|\partial_y j\|_2^{\frac12}\|\partial_{xy} b_2\|_2$$
$$+ C\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_{xy} b_2\|_2^{\frac12}\|\partial_y j\|_2 \le \frac{1}{48}\|\partial_y j\|_2^2 + C\|u_2\|_2^2\|\partial_x u_2\|_2^2\|j\|_2^2.$$
$$J_4 \le \left|\int \partial_x u_1\, \partial_y b_1\, j\right| \le \int\left|(u_2\partial_{yy} b_1 j - u_2\partial_y b_1 \partial_y j)\right| \le C\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|j\|_2^{\frac12}\|\partial_y j\|_2^{\frac12}\|\partial_{yy} b_1\|_2$$
$$+ C\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\partial_y b_1\|_2^{\frac12}\|\partial_{yy} b_1\|_2^{\frac12}\|\partial_y j\|_2 \le \frac{1}{48}\|\partial_y j\|_2^2 + C\|u_2\|_2^2\|\partial_x u_2\|_2^2\|j\|_2^2.$$

To bound J_5, we use $\Delta\cdot u = 0$ and integrate by parts to obtain

$$J_5 = -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega = -\int \partial_x u_1 \omega_x\omega_x - \int \partial_y u_2 \omega_y\omega_y - \int(\partial_x u_2 + \partial_y u_1)\omega_x\omega_y.$$

The terms on the right can be bounded as

$$\int \partial_x u_1 \omega_x\omega_x \le 2\left|\int u_2\,\omega_{xy}\,\omega_x\right| \le C\|\omega_{xy}\|_2\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\omega_x\|_2^{\frac12}\|\omega_{xy}\|_2^{\frac12}$$
$$\le C\|\omega_{xy}\|_2^{\frac32}\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\omega_x\|_2^{\frac12} \le \frac{1}{48}\|\omega_{xy}\|_2^2 + C\|u_2\|_2^2\|\partial_x u_2\|_2^2\|\nabla\omega\|_2^2.$$
$$\left|\int u_1\omega_y\omega_{xy}\right| \le C\|\omega_{xy}\|_2\|u_1\|_2^{\frac12}\|\|\partial_y u_1\|_2^{\frac12}\|\partial_{xy}\omega\|_2^{\frac12}\|\partial_y\omega\|_2^{\frac12} \le C\|\omega_{xy}\|_2^{\frac32}\|u_1\|_2^{\frac12}\|\Omega\|_2^{\frac12}\|\partial_y\omega\|_2^{\frac12}$$
$$\le \frac{1}{48}\|\nabla\omega_y\|_2^2 + C\|u_1\|_2^2\|\partial_y\omega\|_2^2\|\Omega\|_2^2.$$

Furthermore,

$$\left|\int \partial_x u_2\omega_x\omega_y\right| \le \|\partial_x u_2\|_2\|\omega_x\|_2^{\frac12}\|\omega_{xy}\|_2^{\frac12}\|\omega_y\|_2^{\frac12}\|\omega_{xy}\|_2^{\frac12} \le \|\partial_x u_2\|_2\|\nabla\omega\|_2\|\nabla\omega_{xy}\|_2$$
$$\le \frac{1}{48}\|\nabla\omega_y\|_2^2 + C\|\partial_x u_2\|_2^2\|\nabla\omega\|_2^2.$$

The following term can be split into two parts

$$\int \partial_y u_1\omega_x\omega_y = -\int u_1\,\omega_{xy}\,\omega_y - \int u_1\,\omega_x\,\omega_{yy}.$$

Observe that

$$\left|\int u_1\,\omega_{xy}\,\omega_y\right| \le \|\omega_{xy}\|_2\|\omega_y\|_2^{\frac12}\|\omega_{yy}\|_2^{\frac12}\|u_1\|_2^{\frac12}\|\partial_x u_1\|_2^{\frac12} \le \|\nabla\omega_y\|_2^{\frac32}\|\omega_y\|_2^{\frac12}\|u_1\|_2^{\frac12}\|\partial_y u_2\|_2^{\frac12}$$
$$\le \frac{1}{48}\|\nabla\omega_y\|_2^2 + C\|u_1\|_2^2\|\partial_y u_2\|_2^2\|\nabla\omega\|_2^2.$$

$$\left|\int u_1\,\omega_x\omega_{yy}\right| \le \|\omega_{yy}\|_2\|\omega_x\|_2^{\frac{1}{2}}\|\omega_{xy}\|_2^{\frac{1}{2}}\|u_1\|_2^{\frac{1}{2}}\|\partial_x u_1\|_2^{\frac{1}{2}} \le \|\nabla\omega_y\|_2^{\frac{3}{2}}\|\nabla\omega\|_2^{\frac{1}{2}}\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_2\|_2^{\frac{1}{2}}$$
$$\le \frac{1}{48}\|\nabla\omega_y\|_2^2 + C\|u_1\|_2^2\|\partial_y u_2\|_2^2\|\nabla\omega\|_2^2.$$

To estimate J_6, we first integrate by parts to obtain

$$J_6 = 2\int \nabla\omega\cdot\nabla\Omega = -2\int \omega_{xx}\Omega + 2\int \omega_{yy}\Omega.$$

The terms on the right can be bounded as follows.

$$\left|\int \omega_{yy}\Omega\right| \le \|\omega_{yy}\|_2\|\Omega\|_2 \le \frac{1}{48}\|\nabla\omega_y\|_2^2 + C\|\Omega\|_2^2,$$

$$\int \omega_x\Omega_x = \int (\omega_x\partial_{xx}u_2 - \omega_x\partial_{xy}u_1)$$

$$\left|\int \omega_x\partial_{xx}u_2\right| \le C\|\nabla\omega\|_2\|\partial_{xx}u_2\|_2.$$

$$\left|\int \omega_x\partial_{xy}u_1\right| = \left|\int \omega_{xy}\partial_x u_1\right| \le C\|\nabla\omega_y\|_2\|\partial_x u_1\|_2.$$

Combining the estimates above, together with Gronwall's inequalities, we obtain

$$\|\Omega\|_2^2 + \|j\|_2^2 + \|\nabla\omega\|_2^2 + \int_0^t \left(\|\nabla\partial_x u_2, \nabla\partial_y u_2\|_2^2 + \|\partial_y j\|_2^2 + \|\nabla\omega_y\|_2^2 + \|\nabla\omega\|_2^2\right) d\tau \le C$$

for any $t \le T$, where C depends on T and the initial H^1-norm. This completes the proof of Proposition 3.2. □

3.1 | Global H^2 bound and the proof of Theorem 1.1

Proof. It suffices to establish the global H^2-bound in order to prove Theorem 1.1. The rest of this proof establishes the global H^2-bound.

Applying Δ and taking the L^2 inner product of (11) with $\nabla\Omega$ and (12) with ∇j, and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2) + \|\Delta\partial_x u_2\|_2^2 + \|\Delta\partial_y u_2\|_2^2 + \|\nabla\partial_y j\|_2^2 = L_1 + L_2 + L_3 + L_4 + L_5 + L_6, \tag{14}$$

where

$$L_1 = -\int \nabla\Omega\cdot\nabla u\cdot\nabla\Omega\,dxdy, L_2 = -\int \nabla j\cdot\nabla u\cdot\nabla j\,dxdy,$$
$$L_3 = 2\int \nabla\Omega\cdot\nabla b\cdot\nabla j\,dxdy, L_4 = 2\int \nabla[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]\cdot\nabla j\,dxdy,$$
$$L_5 = -2\int \nabla[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\cdot\nabla j\,dxdy, \qquad L_6 = \int \Delta\Omega\,\Delta\omega\,dxdy.$$

Applying Δ to (13) and taking the L^2-inner product with $\Delta\omega$, and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2 + 2\|\Delta\omega_y\|_2^2 + \|\Delta\omega\|_2^2 = \int \Delta\Omega\,\Delta\omega - \int \Delta(u\cdot\nabla\omega)\Delta\omega \equiv L_6 + L_7. \tag{15}$$

Adding (14) and (15) yields

$$\frac{1}{2}\frac{d}{dt}(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2+\|\Delta\omega\|_2^2)+\|\Delta\partial_x u_2\|_2^2+\|\Delta\partial_y u_2\|_2^2+\|\nabla\partial_y j\|_2^2+2\|\Delta\omega_y\|_2^2+\|\Delta\omega\|_2^2$$
$$=L_1+L_2+L_3+L_4+L_5+2L_6+L_7.$$

Because of $\Omega = \Delta \times u$, the following facts are very useful to prove the theorem.

$$\partial_{xx}\Omega=\Delta\partial_x u_2,\quad \partial_{yy}\Omega=-\Delta\partial_y u_1,\quad \partial_{xy}\Omega=\Delta\partial_y u_2.$$

We now estimate L_1 through L_7. We further split L_1 into four terms.

$$L_1=-\int\nabla\Omega\cdot\nabla u\cdot\nabla\Omega\,dxdy=-\int\left(\partial_x u_1(\partial_x\Omega)^2+\partial_x u_2\partial_x\Omega\partial_y\Omega+\partial_y u_1\partial_x\Omega\partial_y\Omega+\partial_y u_2(\partial_y\Omega)^2\right)$$
$$=L_{11}+L_{12}+L_{13}+L_{14}. \tag{16}$$

Now,

$$L_{11}=-\int\partial_x u_1(\partial_{xx}u_2)^2-\int\partial_x u_1(\partial_{xy}u_1)^2+2\int\partial_x u_1\partial_{xx}u_2\partial_{xy}u_1.$$

Integration by parts yields

$$\int\partial_x u_1(\partial_{xx}u_2)^2=-\int\partial_{xx}u_1\partial_{xx}u_2\partial_x u_2-\int\partial_x u_1\partial_{xxx}u_2\partial_x u_2\equiv\quad L_{111}+L_{112},$$

which can be bounded as

$$L_{111}\le C\|\partial_{xx}u_2\|\|\partial_{xx}u_1\|_2^{\frac{1}{2}}\|\partial_{xxy}u_1\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xx}u_2\|_2^{\frac{1}{2}}\le C\|\partial_{xx}u_2\|_2^{\frac{3}{2}}\|\partial_{xx}u_1\|_2^{\frac{1}{2}}\|\Delta\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}$$
$$\le\frac{1}{48}\|\Delta\partial_x u_2\|_2^2+C\|\partial_{xx}u_2\|_2^2\|\partial_{xx}u_1\|_2^{\frac{2}{3}}\|\partial_x u_2\|_2^{\frac{2}{3}}.$$
$$L_{112}\le C\|\partial_{xxx}u_2\|_2\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xx}u_2\|_2^{\frac{1}{2}}\le C\|\Delta\partial_x u_2\|_2\|\Omega\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2\|\partial_x u_2\|_2^{\frac{1}{2}}$$
$$\le\frac{1}{48}\|\Delta\partial_x u_2\|_2^2+C\|\Omega\|_2\|\partial_x u_2\|_2\|\nabla\Omega\|_2^2.$$

By Lemma 1,

$$L_{12}=\int\partial_x u_2\partial_x\Omega\partial_y\Omega\le C\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xy}u_2\|_2^{\frac{1}{2}}\|\partial_y\Omega\|_2\|\partial_x\Omega\|_2^{\frac{1}{2}}\|\partial_{xx}\Omega\|_2^{\frac{1}{2}}\le\frac{1}{48}\|\partial_{xx}\Omega\|_2^2+C\|\Omega\|_2^{\frac{2}{3}}\|\partial_{xx}u_1\|_2^{\frac{2}{3}}\|\nabla\Omega\|_2^2.$$

$$L_{13}\le\left|\int\partial_y u_1\partial_x\Omega\partial_y\Omega\right|\le C\|\partial_y u_1\|_2\|\partial_{xy}\Omega\|_2\|\partial_y\Omega\|_2^{\frac{1}{2}}\|\partial_x\Omega\|_2^{\frac{1}{2}}\le\frac{1}{48}\|\partial_{xy}\Omega\|_2^2+C\|\partial_y u_1\|_2^2\|\nabla\Omega\|_2^2.$$

$$L_{14}\le\left|2\int u_1\partial_y\Omega\partial_{xy}\Omega\right|\le C\|\partial_{xy}\Omega\|_2\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_y\Omega\|_2^{\frac{1}{2}}\|\partial_{xy}\Omega\|_2^{\frac{1}{2}}$$
$$\le C\|\partial_{xy}\Omega\|_2^{\frac{3}{2}}\|u_1\|_2^{\frac{1}{2}}\|\Omega\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\le\frac{1}{48}\|\partial_{xy}\Omega\|_2^2+C\|u_1\|_2^2\|\Omega\|_2^2\|\nabla\Omega\|_2^2.$$

To estimate L_2, we write it out explicitly as

$$L_2=-\int\nabla j\cdot\nabla u\cdot\nabla j\,dxdy$$
$$=-\int\left(\partial_x u_1(\partial_x j)^2+\partial_y u_1\partial_x j\partial_y j+\partial_y u_2(\partial_y j)^2+\partial_x u_2\partial_x j\partial_y j\right)$$
$$=L_{21}+L_{22}+L_{23}+L_{24}.$$

The terms on the right can be bounded as follows.

$$L_{21}=\left|-2\int u_2\partial_x j\partial_{xy}j\right|\le C\|\partial_{xy}j\|_2\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy}j\|_2^{\frac{1}{2}}\|u_2\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}.$$
$$\le\frac{1}{48}\|\nabla\partial_y j\|_2^2+C\|u_2\|_2^2\|\partial_x u_2\|_2^2\|\nabla j\|_2^2.$$
$$L_{22}\le C\|\partial_y u_1\|_2\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_{xy}j\|_2\le\frac{1}{48}\|\nabla\partial_y j\|_2^2+C\|\partial_y u_1\|_2^2\|\nabla j\|_2^2.$$

Similarly,

$$L_{23} \le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|u_1\|_2^2\|\Omega\|_2^2\|\nabla j\|_2^2.$$

$$L_{24} \le C\|\partial_x u_2\|_2\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \le C\|\Omega\|_2\|\nabla j\|_2\|\nabla\partial_y j\|_2.$$

We now turn to L_3. Observe that

$$\begin{aligned} L_3 &= \int \partial_x\Omega\partial_x b_1\partial_x j + \partial_x\Omega\partial_x b_2\partial_y j + \partial_y\Omega\partial_y b_1 j_x + \partial_y\Omega\partial_y b_2\partial_y j \\ &\equiv L_{31} + L_{32} + L_{33} + L_{34}. \end{aligned}$$

The terms on the right can be bounded as follows.

$$\begin{aligned} L_{31} &\le \left|\int \partial_x\Omega\partial_x b_1\partial_x j\right| \le C\|\partial_x\Omega\|_2\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xx} b_1\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \le C\|\partial_x\Omega\|_2\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\le C\|\partial_x j\|_2\|\nabla\partial_y j\|_2 + \|\partial_y j\|\|\partial_x b_1\|\|\partial_x\Omega\|_2^2 \le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_x b_1\|_2^2 + \|\partial_y j\|_2^2 + 1)(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \end{aligned}$$

The last three terms admit,

$$\begin{aligned} L_{32} &\le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_x b_2\|_2^2 + \|\partial_y j\|^2 + 1)(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \\ L_{33} &\le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_x b_1\|_2^2 + \|\partial_y j\|_2^2 + 1)(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \\ L_{34} &\le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_y j\|_2^2 + \|j\|_2 + 1)\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \end{aligned}$$

We now estimate L_4.

$$\begin{aligned} L_4 &= 2\int \nabla[\partial_x b_1(\partial_x u_2 + \partial_y u_1)] \cdot \nabla j \; dxdy \\ &= 2\int \partial_x[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]j_x + \partial_y[\partial_x b_1(\partial_x u_2 + \partial_y u_1)]j_y \; dxdy \\ &\equiv L_{41} + L_{42}. \end{aligned}$$

We bound L_{41} and L_{42} as follows.

$$L_{41} = \int \partial_{xx} b_1\partial_x u_2\partial_x j + \partial_x b_1\partial_{xx} u_2\partial_x j + \partial_{xx} b_1\partial_y u_2\partial_x j + \partial_x b_1\partial_{xy} u_1\partial_x j.$$

Now,

$$\begin{aligned} \left|\int \partial_{xx} b_1\partial_x u_2\partial_x j + \partial_{xx} b_1\partial_y u_2\partial_x j\right| &\le C\|\partial_{xy} b_2\|_2\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xx} u_2\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\quad + C\|\partial_{xy} b_2\|_2\|\partial_y u_2\|_2^{\frac{1}{2}}\|\partial_{xy} u_2\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\le C\|\partial_y j\|_2\|\nabla\partial_y j\|_2^{\frac{1}{2}}\|\partial_x u\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}} \\ &\le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_y j\|_2^2 + \|\partial_x u\|_2^2)(\|\nabla j\|_2^2 + \|\nabla\Omega\|_2^2). \end{aligned}$$

$$\begin{aligned} \left|\int \partial_x b_1\partial_{xx} u_2\partial_x j + \partial_x b_1\partial_{xy} u_1\partial_x j\right| &\le C\|\partial_{xx} u_2\|_2\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xx} b_1\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\quad + C\|\partial_{xy} u_1\|_2\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xx} b_1\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \\ &\le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_x u_1\|_2^2 + \|\partial_x b_1\|_2^2 + \|\partial_y j\|_2^2 + 1)(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2). \end{aligned}$$

L_{42} can be written as

$$L_{42} = 2\int (\partial_{xy} b_1\partial_x u_2 + \partial_x b_1\partial_{xy} u_2 + \partial_{xy} b_1\partial_y u_1 + \partial_x b_1\partial_{yy} u_1)\partial_y j \; dxdy \equiv L_{421} + L_{422} + L_{423} + L_{424}.$$

The bounds for the terms on the right are given as follows.

$$L_{421} \le C\|\partial_x u_2\|_2\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_{xyy} b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}} \le C\|\Omega\|_2\|\nabla j\|_2\|\nabla\partial_y j\|_2 \le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\Omega\|_2^2\|\nabla j\|_2^2.$$

$$L_{422} \le C\|\partial_x b_1\|_2^{\frac12}\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_{xy} u_2\|_2\|\partial_y j\|_2^{\frac12}\|\partial_{xy} j\|_2^{\frac12} \le C\|j\|_2^{\frac12}\|\partial_y j\|_2^{\frac12}\|\partial_x\Omega\|\|\nabla j\|_2^{\frac12}\|\nabla\partial_y j\|_2^{\frac12}$$
$$\le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\partial_y j\|_2^2\|\nabla j\|_2^2 + C\|j\|_2\|\nabla\Omega\|_2^2.$$

Other terms admit,

$$L_{423} \le C\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_{xyy} b_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\partial_y j\|_2 \le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\Omega\|_2\|\nabla j\|_2^2 + C\|\partial_y j\|_2^2\|\nabla\Omega\|_2^2.$$
$$L_{424} \le C\|\partial_y j\|_2\|\partial_x b_1\|_2^{\frac12}\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_{yy} u_1\|_2^{\frac12}\|\partial_{xyy} u_1\|_2^{\frac12} \le \frac{1}{48}\|\partial_{xy}\Omega\|_2^2 + C\|\partial_y j\|_2^2\|\nabla\Omega\|_2^2 + C\|j\|_2\|\nabla j\|_2^2.$$

We now estimate L_5. More explicitly, L_5 can be written as

$$\begin{aligned} L_5 &= -2\int \nabla[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\cdot\nabla j\ dxdy \\ &= -2\int \partial_x[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\partial_x j + \partial_y[(\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\partial_y j\ dxdy \\ &\equiv L_{51} + L_{52}. \end{aligned}$$

L_{52} is bounded as follows.

$$L_{52} \le C\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_{xx} b_2\|_2^{\frac12}\|\partial_{yy} j\|_2 + C\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\partial_y b_1\|_2^{\frac12}\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_{yy} j\|_2$$
$$\le C\|\Omega\|_2^{\frac12}\|\nabla\Omega\|_2^{\frac12}\|j\|_2^{\frac12}\|\nabla j\|_2^{\frac12}\|\nabla\partial_y j\|_2 \le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\Omega\|_2\|j\|_2(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2).$$

L_{51} contains four terms.

$$\begin{aligned} L_{52} &= -2\int(\partial_{xx} u_1\partial_x b_2 + \partial_x u_1\partial_{xx} b_2 + \partial_{xx} u_1\partial_y b_1 + \partial_x u_1\partial_{xy} b_1)\partial_x j\ dxdy \\ &\equiv L_{521} + L_{522} + L_{523} + L_{524}. \end{aligned}$$

These terms are estimated as follows.

$$L_{521} \le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C(\|\partial_y j\|_2^2 + \|j\|_2^2)(\|\nabla j\|_2^2 + \|\nabla\Omega\|_2^2).$$
$$L_{522} \le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\Omega\|_2^{\frac23}\|\partial_{xy} u_1\|_2^{\frac23}\|\nabla j\|_2^2.$$
$$L_{523} \le C\|\partial_{xx} u_1\|_2^{\frac12}\|\partial_{xxy} u_1\|_2^{\frac12}\|\partial_y b_1\|_2^{\frac12}\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_x j\|_2 \le C\|\Omega_{xy}\|_2^{\frac12}\|\partial_y j\|_2^{\frac12}\|\nabla\Omega\|_2^{\frac12}\|j\|_2^{\frac12}\|\nabla j\|_2$$
$$\le \frac{1}{48}\|\partial_{xy}\Omega\|_2^2 + C\|\partial_y j\|_2^2\|\nabla\Omega\|_2^2 + C\|j\|_2\|\nabla j\|_2^2.$$
$$L_{524} \le C\|\partial_x j\|_2\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\partial_{yy} b_1\|_2^{\frac12}\|\partial_{xyy} b_1\|_2^{\frac12} \le C\|\partial_y j\|_2\|\Omega\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\nabla j\|_2^{\frac12}\|\nabla\partial_y j\|_2^{\frac12}$$
$$\le \frac{1}{48}\|\nabla\partial_y j\|_2^2 + C\|\partial_y j\|_2^2\|\nabla\Omega\|_2^2 + \|\Omega\|_2\|\nabla j\|_2^2.$$

L_6 can be easily bounded.

$$L_6 = \int \Delta\Omega\,\Delta\omega = \int \Delta(\partial_x u_2 - \partial_y u_1)\Delta\omega$$

with

$$\int \Delta\partial_x u_2\Delta\omega \le \|\Delta\partial_x u_2\|_2\|\Delta\omega\|_2, \qquad \left|\int \Delta\partial_y u_1\Delta\omega\right| \le \|\Delta u\|_2\|\Delta\omega_y\|_2.$$

We now estimate the last term L_7.

$$\begin{aligned} L_7 &= -\int \Delta(u\cdot\nabla\omega)\Delta\omega = -\int \Delta(u_1\partial_1\omega + u_2\partial_2\omega)\Delta\omega \\ &= -\int \partial_{xx} u_1\partial_x\omega\Delta\omega - \int \partial_{xx} u_2\partial_y\omega\Delta\omega - \int \partial_{yy} u_1\partial_x\omega\Delta\omega - \int \partial_{yy} u_2\partial_y\omega\Delta\omega \\ &\quad - 2\int \partial_x u_1\partial_{xx}\omega\Delta\omega - 2\int \partial_x u_2\partial_{xy}\omega\Delta\omega - 2\int \partial_y u_1\partial_{xy}\omega\Delta\omega - 2\int \partial_y u_2\partial_{yy}\omega\Delta\omega \\ &\equiv L_{71} + L_{72} + L_{73} + L_{74} + L_{75} + L_{76} + L_{77} + L_{78}. \end{aligned}$$

Now,

$$L_{71}\left|\leq \int \partial_{xx}u_1\partial_x\omega\Delta\omega\right| \leq C\|\partial_{xx}u_1\|_2\|\partial_x\omega\|_2^{\frac{1}{2}}\|\partial_{xx}\omega\|_2^{\frac{1}{2}}\|\Delta\omega\|_2^{\frac{1}{2}}\|\Delta\partial_y\omega\|_2^{\frac{1}{2}}$$
$$\leq \frac{1}{48}\|\Delta\partial_y\omega\|_2^2 + C(\|\nabla\omega\|_2^2 + \|\partial_{xx}u_1\|_2^2)\|\Delta\omega\|_2^2.$$

Similarly, we obtain

$$L_{72} \leq \frac{1}{48}(\|\Delta\partial_y\omega\|_2^2 + \|\Delta\partial_x u_2\|_2^2) + C\|\nabla\omega\|_2^2(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2).$$

$$L_{73} = \left|-\int \partial_u u_1\partial_{xy}\omega\Delta\omega - \int \partial_y u_1\partial_x\omega\Delta\partial_y\omega\right|$$
$$\leq C\|\partial_{xy}\omega\|_2\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\Delta\omega\|_2^{\frac{1}{2}}\|\Delta\partial_y\omega\|_2^{\frac{1}{2}} + C\|\Delta\partial_y\omega\|_2\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_x\omega\|_2^{\frac{1}{2}}\|\partial_{xy}\omega\|_2^{\frac{1}{2}}$$
$$\leq C\|\nabla\partial_y\omega\|_2\|\partial_y u_1\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\|\Delta\omega\|_2^{\frac{1}{2}}\|\Delta\partial_y\omega\|_2^{\frac{1}{2}} + C\|\Delta\partial_y\omega\|_2\|\partial_y u_1\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\|\partial_x\omega\|_2^{\frac{1}{2}}\|\nabla\partial_y\omega\|_2^{\frac{1}{2}}$$
$$\leq \frac{1}{48}\|\Delta\partial_y\omega\|_2^2 + C(\|\partial_y u_1\|_2^2 + \|\nabla\partial_y\omega\|_2^2 + 1)(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2 + \|\nabla\omega\|_2^2).$$

Similarly,

$$L_{74} \leq \frac{1}{48}(\|\Delta\partial_y\omega\|_2^2 + \|\Delta\partial_x u_2\|_2^2) + C\|\nabla\omega\|_2^2(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2).$$
$$L_{75} \leq \frac{1}{48}\|\Delta\partial_y\omega\|_2^2 + C(\|\Omega\|_2^2 + \|\partial_{xx}u_1\|_2^2)\|\Delta\omega\|_2^2.$$

$$L_{76} = \left|-2\int \partial_x u_2\partial_{xy}\omega\Delta\omega\right| \leq C\|\partial_{xy}\omega\|_2\|\Delta\omega\|_2^{\frac{1}{2}}\|\Delta\partial_y\omega\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}|\partial_{xx}u_2\|_2^{\frac{1}{2}}$$
$$\leq C\|\nabla\partial_y\omega\|_2\|\Delta\omega\|_2^{\frac{1}{2}}\|\Delta\partial_y\omega\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}$$
$$\leq \frac{1}{48}\|\Delta\partial_y\omega\|_2^2 + C(\|\nabla\partial_y\omega\|_2^2 + \|\partial_x u_2\|_2^2)(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2).$$

$$L_{77} = \left|-2\int \partial_y u_1\partial_{xy}\omega\Delta\omega\right| \leq C\|\partial_{xy}\omega\|_2\|\Delta\omega\|_2^{\frac{1}{2}}\|\Delta\partial_y\omega\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}$$
$$\leq C\|\nabla\partial_y\omega\|_2\|\Delta\omega\|_2^{\frac{1}{2}}\|\Delta\partial_y\omega\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}$$
$$\leq \frac{1}{48}\|\Delta\partial_y\omega\|_2^2 + C(\|\nabla\partial_y\omega\|_2^2 + \|\partial_y u_1\|_2^2)(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2).$$

Finally, note that

$$L_{78} \leq \frac{1}{48}\|\Delta\partial_y\omega\|_2^2 + C(\|\Omega\|_2^2 + \|\partial_{xx}u_1\|_2^2)\|\Delta\omega\|_2^2.$$

Collecting the estimates above and applying Gronwall's inequality, we obtain the desired global H^2-bound. This completes the proof for the global H^2-bound. □

4 | PROOF OF PROPOSITION 1.2

In this section, we prove proposition 1.2. We can easily prove the following L^2-bound.

Lemma 4.1. *Assume that (u_0, b_0, ω_0) satisfies the condition stated in Theorem 1.2. Let (u, b, ω) be the corresponding solution of (6). Then, for any $T > 0$, (u, b, ω) obeys the following global L^2-bound,*

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 + 2\int_0^T \|\partial_x u_2\|_{L^2}^2 d\tau + 2\int_0^T \|\partial_y b(\tau)\|_{L^2}^2 d\tau + 2\int_0^T \|\partial_x\omega(\tau)\|_{L^2}^2 d\tau \leq C(\|(u_0, b_0, \omega_0)\|_2^2).$$

Proof. To estimate the H^1-norm of (u, b, ω), we consider the equations of $\Omega = \Delta \times u$, $\Delta\omega$, and of the current density $j = \Delta \times b$,

$$\Omega_t + u\cdot\nabla\Omega = \partial_{xxx}u_2 + (b\cdot\nabla)j - \Delta\omega, \tag{17}$$

$$j_t + u\cdot\nabla j = -\partial_{yyy}b_1 + \partial_{xyy}b_2 + b\cdot\nabla\Omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \tag{18}$$

$$\partial_t\nabla\omega + \nabla(u\cdot\nabla\omega) + \nabla\omega = \nabla\omega_{xx} + \nabla\Omega, \tag{19}$$
$$\nabla\cdot u = 0, \quad \nabla\cdot b = 0.$$

Dotting (17) by Ω, (18) by j, and (19) by $\Delta\omega$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2+\|j\|_{L^2}^2+\|\nabla\omega\|_2^2\right)+\|\partial_{xx}u_1\|_2^2+\|\partial_{xx}u_2\|_2^2+\|\partial_y j\|_{L^2}^2+\|\nabla\omega_x\|_2^2+2\|\nabla\omega\|_2^2$$
$$=2\int\left[\partial_x b_1(\partial_x u_2+\partial_y u_1)j-\partial_x u_1(\partial_x b_2+\partial_y b_1)\right]j\,dx\,dy-\int\nabla\omega\cdot\nabla u\cdot\nabla\omega+\int\nabla\omega\cdot\nabla\Omega$$
$$\equiv J_1+J_2+J_3+J_4+J_5+J_6.$$

Invoking the divergence-free condition, we

$$\|\nabla\partial_x u_2\|_2^2=\|\partial_{xx}u_1\|_2^2+\|\partial_{xx}u_2\|_2^2.$$

We now estimate the terms on the right. Since $j=\partial_x b_2-\partial_y b_1$,

$$J_1=2\int\partial_x b_1\partial_x u_2\partial_x b_2-2\int\partial_x b_1\partial_x u_2\partial_y b_1\equiv J_{11}+J_{12}.$$

Applying Lemma 1, Young's inequality, and the simple fact that

$$\|\partial_x b_2\|_{L^2}\le\|j\|_{L^2},\quad \|\partial_{xy}b_1\|_{L^2}\le\|\partial_y j\|_{L^2},$$

we have

$$J_{11}\le 2\left|\int\partial_x b_1\partial_x u_2\partial_x b_2\right|\le C\|\partial_x b_2\|_2\|\partial_x u_2\|_2^{\frac12}\|\partial_{xx}u_2\|_2^{\frac12}\|\partial_x b_1\|_2^{\frac12}\|\partial_{xy}b_1\|_2^{\frac12}\le C\|j\|_2\|\partial_x u_2\|_2^{\frac12}\|\partial_{xx}u_2\|_2^{\frac12}\|\partial_y b_2\|_2^{\frac12}\|\partial_y j\|_2^{\frac12}$$
$$\le\|\partial_{xx}u_2\|_2\|\partial_y j\|_2+C\|\partial_y b_2\|_2\|\partial_x u_2\|_2\|j\|_2^2\le\frac{1}{48}(\|\partial_{xx}u_2\|_2^2+\|\partial_y j\|_2^2)+C(\|\partial_y b\|_2^2+\|\partial_x u_2\|_2^2)\|j\|_2^2.$$

Integrating by parts, we have

$$J_{12}\le\left|2\int\partial_x b_1\partial_x u_2\partial_y b_1\right|\le C\|\partial_y b_2\|_2\|\partial_x u_2\|_2^{\frac12}\|\partial_{xx}u_2\|_2^{\frac12}\|\partial_y b_1\|_2^{\frac12}\|\partial_{yy}b_1\|_2^{\frac12}$$
$$\le\frac{1}{48}(\|\partial_{xx}u_2\|_2^2+\|\partial_{yy}b_1\|_2^2)+C(\|\partial_x u_2\|_2^2+\|\partial_y b_1\|_2^2)\|j\|_2^2.$$

The dissipation is not sufficient to control J_2, this is the place where Lemma 1 cannot apply. J_2 can be bounded as

$$J_2\le\left|\int\partial_x b_1\,\partial_y u_1\,j\right|\le\|\partial_y u_1\|_{BMO}\|\partial_x b_1\|_2\|j\|_2.$$

J_3, and J_4 can be bounded by

$$J_3\le\left|\int\partial_x u_1\,\partial_x b_2\,j\right|\le\int\left|(u_2\partial_{xy}b_2 j+u_2\partial_x b_2\partial_y j)\right|$$
$$\le C\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|j\|_2^{\frac12}\|\partial_y j\|_2^{\frac12}\|\partial_{xy}b_2\|_2+C\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_{xy}b_2\|_2^{\frac12}\|\partial_y j\|_2$$
$$\le\frac{1}{48}\|\partial_y j\|_2^2+C\|u_2\|_2^2\|\partial_x u_2\|_2^2\|j\|_2^2.$$

Similarly,

$$J_4\le\frac{1}{48}\|\partial_y j\|_2^2+C\|u_2\|_2^2\|\partial_x u_2\|_2^2\|j\|_2^2.$$

To bound J_5, we use $\Delta\cdot u=0$ and integrate by parts to obtain

$$J_5=-\int\nabla\omega\cdot\nabla u\cdot\nabla\omega=-\int\partial_x u_1\omega_x\omega_x-\int\partial_y u_2\omega_y\omega_y-\int(\partial_x u_2+\partial_y u_1)\omega_x\omega_y.$$

The terms on the right can be bounded as

$$\int\partial_x u_1\omega_x\omega_x\le 2\left|\int u_2\,\omega_{xy}\,\omega_x\right|\le C\|\omega_{xy}\|_2\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\omega_x\|_2^{\frac12}\|\omega_{xy}\|_2^{\frac12}$$
$$\le C\|\omega_{xy}\|_2^{\frac32}\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\omega_x\|_2^{\frac12}\le\frac{1}{48}\|\omega_{xy}\|_2^2+C\|u_2\|_2^2\|\partial_x u_2\|_2^2\|\nabla\omega\|_2^2.$$

$$\left|\int \partial_y u_2 \omega_y \omega_y\right| \le \|\partial_y u_2\|_{BMO}\|\nabla\omega\|_2^2$$

$$\left|\int \partial_x u_2 \omega_x \omega_y\right| \le \|\partial_x u_2\|_2 \|\omega_x\|_2^{\frac{1}{2}}\|\omega_{xy}\|_2^{\frac{1}{2}}\|\omega_y\|_2^{\frac{1}{2}}\|\omega_{xy}\|_2^{\frac{1}{2}} \le C\|\partial_x u_2\|_2\|\nabla\omega\|_2\|\nabla\omega_x\|_2$$
$$\le \frac{1}{48}\|\nabla\omega_x\|_2^2 + C\|\partial_x u_2\|_2^2\|\nabla\omega\|_2^2.$$

$$\left|\int \partial_y u_1 \omega_x \omega_y\right| \le C\|\partial_y u_1\|_{BMO}\|\omega_x\|_2\|\omega_y\|_2.$$

To estimate J_6, we first integrate by parts and obtain

$$J_6 = 2\int \nabla\omega\cdot\nabla\Omega = -2\int \omega_{xx}\Omega + 2\int \omega_{yy}\Omega.$$

The terms on the right can be bounded as follows.

$$\left|\int \omega_{xx}\Omega\right| \le \|\omega_{xx}\|_2\|\Omega\|_2 \le \frac{1}{48}\|\nabla\omega_x\|_2^2 + C\|\Omega\|_2^2,$$
$$\int \omega_{yy}\Omega = \int \omega_y\Omega_y = \int (\omega_y\partial_{xy}u_2 - \omega_y\partial_{yy}u_1),$$
$$\left|\int \omega_y\partial_{xy}u_2\right| \le C\|\nabla\omega\|_2\|\partial_{xx}u_1\|_2,$$
$$\left|\int \omega_y\partial_{yy}u_1\right| = \left|\int \omega_{xy}\partial_y u_1\right| \le \|\partial_y u_1\|_{BMO}\|\nabla\omega_y\|_2.$$

Combining the estimates above, together with Gronwall's inequalities, we obtain

$$\|\Omega\|_2^2 + \|j\|_2^2 + \|\nabla\omega\|_2^2 + \int_0^t \left(\|\partial_{xx}u_1\|_2^2 + \|\partial_{xx}u_2\|_2^2 + \|\partial_y j\|_{L^2}^2 + \|\nabla\omega_x\|_2^2 + \|\nabla\omega\|_2^2\right) d\tau \le C$$

for any $t \le T$, where C depends on T and the initial H^1-norm. This completes the proof of proposition 1.2. □

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CONFLICT OF INTERESTS

There are no conflict of interests to this work.

ORCID

Dipendra Regmi https://orcid.org/0000-0001-8439-8295

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