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A regularity criterion for two-and-half-dimensional magnetohydrodynamic equations with horizontal dissipation and horizontal magnetic diffusion

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We study the global regularity of classical solution to two-and-half-dimensional magnetohydrodynamic equations with horizontal dissipation and horizontal magnetic diffusion. We prove that any possible finite time blow-up can be controlled by the L^∞-norm of the vertical components. Copyright © 2016 John Wiley & Sons, Ltd.

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1. Introduction

The magnetohydrodynamic (MHD) equations govern the dynamics of the velocity and the magnetic field in electrically conducting fluids and reflect the basic physics laws of conservation. The MHD equations involve coupling between the Navier–Stokes equations governing the fluid and the Maxwell's equations governing the magnetic field. One of the most fundamental problems in fluid dynamics concerning the MHD equations is whether their classical solutions are globally regular for all time or they develop singularities. The MHD equations have been a center of attention to numerous analytical, experimental, and numerical investigations.

The two-dimensional MHD equations can be written as

$$\begin{cases} u_t + u\cdot\nabla u = -\nabla p + \nu_1\, u_{xx} + \nu_2\, u_{yy} + b\cdot\nabla b, \\ b_t + u\cdot\nabla b = \eta_1\, b_{xx} + \eta_2\, b_{yy} + b\cdot\nabla u, \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0, \\ u(x,y,0) = u_0(x,y),\ b(x,y,0) = b_0(x,y), \end{cases} \tag{1.1}$$

where $(x,y) \in \mathbb{R}^2$, $t \geq 0$, $u = (u_1(x,y,t),\ u_2(x,y,t))$ denotes the two-dimensional velocity field, $p = p(x,y,t)$ the pressure, $b = (b_1(x,y,t), b_2(x,y,t))$ the magnetic field, and ν_1, ν_2, η_1, and η_2 are nonnegative real parameters. When $\nu_1 = \nu_2$ and $\eta_1 = \eta_2$, (1.1) reduces to the standard incompressible MHD equations.

When all four parameters ν_1, ν_2, η_1, and η_2 are positive, the global regularity of two-dimensional MHD equations is well known [1]. However, it remains a remarkable open problem whether classical solutions of the two-dimensional inviscid MHD equation, all four parameters are zero, preserve their regularity for all time or finite time blow-up. Many attempts have been made, but there are no satisfactory results concerning the regularity of the solution. When $\nu_1 > 0$, $\nu_2 = 0$, $\eta_1 = 0$, and $\eta_2 > 0$ or when $\nu_1 = 0$, $\nu_2 > 0$, $\eta_1 > 0$, and $\eta_2 = 0$, the global regularity was recently established by Cao and Wu [2].

Cao, Regmi, and Wu studied two-dimensional MHD equations with horizontal dissipation and horizontal diffusion in [3]. They proved that any possible blow-up can be controlled by the L^∞-norm of the horizontal components. Furthermore, they prove that L^{2r}-norm of horizontal components, $1 < r < \infty$, cannot grow faster than $\sqrt{r\log r}$ as r increases.

It is currently unknown whether the solutions of the three-dimensional MHD equations is globally regular (in time). There are numerous papers related to the global regularity of three-dimensional MHD equations (see [1, 4–16] and references therein).

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The two-dimensional flow generates a large family of three-dimensional flow with vorticity stretching([17]); we refer to these as two-and-half-dimensional flows because the flow in the z direction is predetermined by the underlying two-dimensional flows. Very recently, Ji and Li ([18]) studied the global regularity of two-and-half-dimensional MHD equations. They proved the global regularity (in time) for two-and-half-dimensional MHD equations with mixed dissipation diffusion. Furthermore, they established a conditional global regularity for the horizontal dissipation and horizontal magnetic diffusion. One natural question is 'Can we extend the results related to two-dimensional MHD equations to two-and-half-dimensional MHD equations?'

We study two-and-half-dimensional MHD (i.e., (u, b) and p are independent of z). Let $u = (u_1, u_2, u_3) = (\tilde{u}, u_3)$, $b = (b_1, b_2, b_3) = (\tilde{b}, b_3)$, then two-and-half-dimensional MHD equations can be written as

$$\begin{cases} \tilde{u}_t + (\tilde{u}\cdot\tilde{\nabla})\tilde{u} = -\tilde{\nabla}p + \nu_1\tilde{u}_{xx} + \nu_2\tilde{u}_{yy} + (\tilde{b}\cdot\tilde{\nabla})\tilde{b}, \\ \partial_t u_3 + (\tilde{u}\cdot\tilde{\nabla})u_3 = (\tilde{b}\cdot\tilde{\nabla})u_3 + \nu_1\partial_{xx}u_3 + \nu_2\partial_{yy}u_3, \\ \tilde{b}_t + (\tilde{u}\cdot\tilde{\nabla})\tilde{b} = \eta_1\partial_{xx}\tilde{b} + \eta_2\partial_{yy}\tilde{b} + (\tilde{b}\cdot\tilde{\nabla})\tilde{u}, \\ \partial_t b_3 + (\tilde{u}\cdot\tilde{\nabla})b_3 = (\tilde{b}\cdot\tilde{\nabla})b_3 + \eta_1\partial_{xx}b_3 + \eta_2\partial_{yy}b_3, \\ \tilde{\nabla}\cdot\tilde{u} = 0, \quad \tilde{\nabla}\cdot\tilde{b} = 0, \\ u(x,y,0) = u_0(x,y), \ b(x,y,0) = b_0(x,y), \end{cases} \tag{1.2}$$

where $u : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^3$ denotes velocity field, $b : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}^3$ magnetic field, and $p : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ pressure.

The vorticity $\omega = (\partial_y u_3, -\partial_x u_3, \partial_x u_2 - \partial_y u_1)$ and current density $j = (\partial_y b_3, -\partial_x b_3, \partial_x b_2 - \partial_y b_1)$ for two-and-half-dimensional MHD equations can be written as

$$\begin{cases} \partial_t\omega_1 + (\tilde{u}\cdot\tilde{\nabla})\omega_1 = (\omega\cdot\nabla)u_1 + (\tilde{b}\cdot\tilde{\nabla})j_1 - (j\cdot\nabla)b_1 + \nu_1\partial_{xx}\omega_1 + \nu_2\partial_{yy}\omega_1 \\ \partial_t\omega_2 + (\tilde{u}\cdot\tilde{\nabla})\omega_2 = (\omega\cdot\nabla)u_2 + (\tilde{b}\cdot\tilde{\nabla})j_2 - (j\cdot\nabla)b_2 + \nu_1\partial_{xx}\omega_2 + \nu_2\partial_{yy}\omega_2 \\ \partial_t\omega_3 + (\tilde{u}\cdot\tilde{\nabla})\omega_3 = (\tilde{b}\cdot\tilde{\nabla})j_3 + \nu_1\partial_{xx}\omega_3 + \nu_2\partial_{yy}\omega_3 \\ \partial_t j_1 + (\tilde{u}\cdot\tilde{\nabla})j_1 = (j\cdot\nabla)u_1 + (\tilde{b}\cdot\tilde{\nabla})\omega_1 - (\omega\cdot\nabla)b_1 + \eta_1\partial_{xx}j_1 + \eta_2\partial_{yy}j_2 \\ \partial_t j_2 + (\tilde{u}\cdot\tilde{\nabla})j_2 = (j\cdot\nabla)u_2 + (\tilde{b}\cdot\nabla)\omega_2 - (\omega\cdot\nabla)b_2 + \eta_1\partial_{xx}j_2 + \eta_2\partial_{yy}j_2 \\ \partial_t j_3 + (\tilde{u}\cdot\tilde{\nabla}j_3 = (\tilde{b}\cdot\tilde{\nabla})\omega_3 + \partial_{xx}j_3 + 2\partial_1 b_1(\partial_y u_1 + \partial_x u_2) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1) \end{cases} \tag{1.3}$$

Our main goal in this paper is to study the global regularity of the following two-and-half-dimensional MHD equations with horizontal dissipation and horizontal diffusion:

$$\begin{cases} \tilde{u}_t + (\tilde{u}\cdot\tilde{\nabla})\tilde{u} = -\tilde{\nabla}p + \nu_1\tilde{u}_{xx} + (\tilde{b}\cdot\tilde{\nabla})\tilde{b}, \\ \partial_t u_3 + (\tilde{u}\cdot\tilde{\nabla})u_3 = (\tilde{b}\cdot\tilde{\nabla})u_3 + \nu_1\partial_{xx}u_3, \\ \tilde{b}_t + (\tilde{u}\cdot\tilde{\nabla})\tilde{b} = \eta_1\partial_{xx}\tilde{b} + (\tilde{b}\cdot\tilde{\nabla})\tilde{u}, \\ \partial_t b_3 + (\tilde{u}\cdot\tilde{\nabla})b_3 = (\tilde{b}\cdot\tilde{\nabla})b_3 + \eta_1\partial_{xx}b_3, \\ \tilde{\nabla}\cdot\tilde{u} = 0, \quad \tilde{\nabla}\cdot\tilde{b} = 0, \\ u(x,y,0) = u_0(x,y), \ b(x,y,0) = b_0(x,y) \end{cases} \tag{1.4}$$

For simplicity, we take $\eta_1 = \nu_1 = 1$ in the calculation. $\|f\|_p$ denotes L^p-norm of $f \in L^p(\mathbb{R}^2)$.

In this paper, we prove the following theorem.

Theorem 1.1

Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$. Let (u, b) be the solution of (1.4). If

$$\int_0^T \|(u_1, b_1)\|_\infty^2 dt < \infty$$

for some $T > 0$, then $\|(u, b)\|_{H^2}$ is finite on $[0, T]$.

Similarly, we can prove

Theorem 1.2

Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla\cdot u_0 = 0$ and $\nabla\cdot b_0 = 0$. Let (u, b) be the solution of (1.2) with $\nu_1 = 0$, $\nu_2 > 0$ $\eta_1 = 0$, $\eta_2 > 0$. If

$$\int_0^T \|(u_2, b_2)\|_\infty^2 dt < \infty$$

for some $T > 0$, then $\|(u, b)\|_{H^2}$ is finite on $[0, T]$.

Remark 1.3

(1) Ji and Li ([18]) established a conditional global regularity for the horizontal dissipation and horizontal magnetic diffusion, which is $\|\partial_y u_1\|_{L^2((0,T);\mathbb{R}^2)} < \infty$ or $\|\partial_y b_1\|_{L^2((0,T);\mathbb{R}^2)} < \infty$

(2) Our method is similar to two-dimensional MHD equations with horizontal dissipation and horizontal diffusion [3]. However, in the presence of the vortex stretching term, the mathematical analysis for two-and-half-dimensional is harder than two-dimensional case.

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The rest of the paper is organized as follows: In Section 2, first, we state key lemmas and then prove the global H^1 bound for (u, b). The last section presents the proof of the main theorem.

2. Global H^1-bound for (u, b)

One can easily prove the following L^2-bound for (u, b).

Lemma 2.1
Let $(u_0, b_0) \in H^2(\mathbb{R}^2)$ and let (u, b) be the corresponding solution of (1.4). Then, (u, b) obeys the following global L^2-bound:

$$\|u(t)\|_2^2 + \|b(t)\|_2^2 + 2\int_0^t \|\partial_x u(\tau)\|_2^2\, d\tau + 2\int_0^t \|\partial_x b(\tau)\|_2^2\, d\tau \le \|u_0\|_2^2 + \|b_0\|_2^2 \tag{2.1}$$

for any $t \ge 0$.

We need the anisotropic Sobolev inequalities stated in the following lemma [2].

Lemma 2.2
If $f, g, h, \partial_y g, \partial_x h \in L^2(\mathbb{R}^2)$, then

$$\iint_{\mathbb{R}^2} |f\, g\, h|\, dxdy \le C\, \|f\|_2\, \|g\|_2^{\frac{1}{2}}\, \|\partial_y g\|_2^{\frac{1}{2}}\, \|h\|_2^{\frac{1}{2}}\, \|\partial_x h\|_2^{\frac{1}{2}} \tag{2.2}$$

where C is a constant.

Next step is to prove H^1 bound for (u, b). More precisely, we prove the following theorem.

Theorem 2.3
Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$. Let (u, b) be the corresponding solution of (1.4). Then, for any $T > 0$ and $t \le T$,

$$\|(u(t),\, b(t))\|_{H^1} \le C_1 e^{C_2 \int_0^t \left(\|u_1\|_\infty^2 + \|b_1\|_\infty^2\right) d\tau}$$

where C_1 is a constant depending on T and initial data and C_2 is a pure constant.

We prove Theorem 2.3 in two steps. First, we obtain the global L^2 bound for ω_3 and j_3, and then by employing the bound of $\|\omega_3\|_2$ and $\|j_3\|_2$, we prove the global L^2-bound for $\omega_1, \omega_2, j_1, j_2$.

2.1. L^2 bound for (ω_3, j_3)

Proposition 2.4
Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$ and let (u, b) be the solution of (1.4). Then, ω_3 and j_3 satisfy

$$\|\omega_3\|_2^2 + \|j_3\|_2^2 + \int_0^T \|\partial_x \omega_3\|_2^2 + \|\partial_x j_3\|_2^2 \le C$$

if $(u_1, b_1) \in L^2([0, T]; L^\infty(\mathbb{R}^2))$.

Proof of (2.4): We consider the case $\nu_1 > 0$, $\nu_2 = 0$ $\eta_1 > 0$, $\eta_2 = 0$ in (1.3). Taking an inner product of the third and sixth equations in (1.3) with ω_3 and j_3, respectively, integrating with respect to space

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\omega_3\|_{L^2}^2 + \|j_3\|_{L^2}^2\right) + \|\partial_x \omega_3\|_{L^2}^2 + \|\partial_x j_3\|_{L^2}^2 \\ &\quad = 2\int \partial_x b_1(\partial_x u_2 + \partial_y u_1)\, j_3\, dxdy - 2\int \partial_x u_1(\partial_x b_2 + \partial_y b_1)\, j_3\, dxdy. \\ &\quad = J_1 + J_2 + J_3 + J_4 \end{aligned} \tag{2.3}$$

For notational convenience, set

$$Y(t) = \|\omega_3(\cdot, t)\|_{L^2}^2 + \|j_3(\cdot, t)\|_{L^2}^2$$

The first term can be bounded by Lemma 2.2, namely,

$$\begin{aligned} J_1 + J_3 \le{}& C\, \|\partial_x b_1\|_2^{\frac{1}{2}}\, \|\partial_{xy} b_1\|_2^{\frac{1}{2}}\, \|\partial_x u_2\|_2^{\frac{1}{2}}\, \|\partial_{xx} u_2\|_2^{\frac{1}{2}}\, \|j_3\|_2 + \\ & C\|j_3\|_2 \|\partial_y u_2\|_2^{\frac{1}{2}}\, \|\partial_{xy} u_2\|_2^{\frac{1}{2}}\, \|\partial_x b_2\|_2^{\frac{1}{2}}\, \|\partial_{xy} b_2\|_2^{\frac{1}{2}} \end{aligned}$$

we have

$$J_1 + J_3 \le \frac{1}{8}\|\partial_x \omega_3\|_2^2 + \frac{1}{8}\|\partial_x j_3\|_2^2 + C\, \|\partial_x u, \partial_x b\|_2^2\, Y(t).$$

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The second term on the right of (2.3) needs to be handled differently. By integration by parts,

$$J_2 \equiv 2\int \partial_x b_1\, \partial_y u_1\, j_3 = -2\int b_1 \partial_x \partial_y u_1\, j_3 - 2\int b_1 \partial_y u_1\, \partial_x j_3 \equiv J_{21} + J_{22}. \tag{2.4}$$

$$J_{21} \le \|b_1\|_\infty \|\partial_x j_3\|_2 \|j_3\|_2 \le \frac{1}{8}\|\partial_x j_3\|_2^2 + \|b_1\|_\infty^2 \|j_3\|_2^2$$
$$J_{22} \le \|b_1\|_\infty \|\partial_y u_1\|_2 \|\partial_x j_3\|_2 \le \frac{1}{8}\|\partial_x j_3\|_2^2 + \|b_1\|_\infty^2 \|\omega_3\|_2^2$$

Thus,

$$J_2 \le \frac{1}{4}\|\partial_x j_3\|_2^2 + \|b_1\|_\infty^2 Y(t)$$
$$J_4 = -2\int \partial_x u_1 \partial_y b_1 \partial_x b_2 = -2\int \partial_x u_1 \partial_y b_1 \partial_y b_1 = J_{41} + J_{42}$$

J_{41} admits

$$J_{41} \le \frac{1}{8}\|\partial_x \omega_3\|_2^2 + \frac{1}{8}\|\partial_x j_3\|_2^2 + C\|b\|_2^2 \|\partial_x b\|_2^2 \left(\|\omega_3\|_2^2 + \|j_3\|_2^2\right)$$
$$J_{42} = -4\int u_1 \partial_y b_1 \partial_{xy} b_1 \le \frac{1}{8}\|\partial_x j_3\|_2^2 + \frac{1}{8}\|u_1\|_\infty^2 \|j_3\|_2^2$$

Thus,

$$J_4 \le \frac{1}{4}\|\partial_x \omega_3, \partial_x j_3\|_2^2 + C\|u_1\|_\infty^2 Y(t)$$

After combining all inequalities,

$$\frac{1}{2}\frac{d}{dt}Y(t) + \|\partial_x \omega_3\|_{L^2}^2 + \|\partial_x j_3\|_{L^2}^2 \le \frac{1}{2}\|\partial_x \omega_3\|_2^2 + \frac{1}{2}\|\partial_x \omega_3\|_2^2 + C\left(\|u_1\|_\infty^2 + \|b_1\|_\infty^2\right) Y(t)$$

After applying Gronwall's lemma, we get L^2-bound for ω_3 and j_3.

2.2. L^2 bound for $(\omega_1, \omega_2, j_1, j_2)$

Proposition 2.5
Assume that $(u_0, b_0) \in H^2(\mathbb{R}^2)$, $\nabla \cdot u_0 = 0$ and $\nabla \cdot b_0 = 0$ and let (u, b) be the solution of (1.4). Then, (ω_1, ω_2) and (j_1, j_2) satisfy

$$\|(\omega_1, \omega_2)\|_2^2 + \|(j_1, j_2)\|_2^2 + \int_o^T \|(\partial_x \omega_1, \partial_x \omega_2)\|_2^2 + \|(\partial_x j_1, \partial_x j_2)\|_2^2 \le C$$

if $(u_1, b_1) \in L^2([0, T]; L^\infty(\mathbb{R}^2))$.

Proof of 2.5. We consider the case $\nu_1 > 0$, $\nu_2 = 0$ $\eta_1 > 0$, $\eta_2 = 0$ in (1.3). Taking an inner product of the first, second, fourth, and fifth equations in (1.3) with ω_1, ω_1, j_1, and j_2, respectively, integrating with respect to space variable

$$\frac{1}{2}\frac{d}{dt}\left[\|(\omega_1, \omega_2)\|_2^2 + \|(j_1, j_2)\|_2^2\right] + \|(\partial_x \omega_1, \partial_x \omega_2)\|_2^2 + \|(\partial_x j_1, \partial_x j_2)\|_2^2 == \sum_{i=1}^{8} K_i$$

where

$$K_1 = \int (\omega \cdot \nabla) u_1\, \omega_1\, dx\, dy, \qquad K_2 = -\int (j \cdot \nabla) b_1\, \omega_1\, dx\, dy$$
$$K_3 = \int (\omega \cdot \nabla) u_2 \omega_2\, dx\, dy, \qquad K_4 = -\int (j \cdot \nabla) b_2 \omega_2\, dx\, dy$$
$$K_5 = \int (j \cdot \nabla) u_1 j_1\, dx\, dy, \qquad K_6 = -\int (\omega \cdot \nabla) b_1 j_1\, dx\, dy$$
$$K_7 = \int (j \cdot \nabla) u_2 j_2\, dx\, dy, \qquad K_8 = -\int (\omega \cdot \nabla) b_2 j_2\, dx\, dy$$

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For notational convenience, set $Z(t) = \|(\omega_1, \omega_2)\|_2^2 + \|(j_1, j_2)\|_2^2$. Now,

$$\begin{aligned} K_1 &= \int \omega_1 \partial_x u_1 \, \omega_1 + \omega_2 \partial_y u_1 \omega_1 = -2 \int u_1 \partial_x \omega_1 \, \omega_1 + \int \omega_2 \partial_y u_1 \, \omega_1 \\ &\leq \|u_1\|_\infty \|\partial_x \omega_1\|_2 \|\omega_1\|_2 + \|\partial_y u_1\|_2 \|\omega_2\|_2^{\frac{1}{2}} \|\partial_y \omega_2\|_2^{\frac{1}{2}} \|\omega_1\|_2^{\frac{1}{2}} \|\partial_x \omega_1\|_2^{\frac{1}{2}} \\ &\leq \frac{1}{16} \|\partial_x \omega_1\|_2^2 + \|u_1\|_\infty^2 \|\omega_1\|_2^2 + \|\partial_y u_1\|_2^2 \left[\|\omega_1\|_2^2 + \|\omega_2\|_2^2 \right] \\ &\leq \frac{1}{16} \|\partial_x \omega_1\|_2^2 + \left(\|u_1\|_\infty^2 + \|\partial_y u_1\|_2^2 \right) Z(t) \end{aligned}$$

Because $\partial_y \omega_2 = -\partial_x \omega_1$ and $\partial_y u_1 = \partial_x u_2 - \omega_3$, so $\int \|\partial_y u_1\|_2^2 < \infty$. Similarly, $\int \|\partial_y b_1\|_2^2 < \infty$

$$\begin{aligned} K_2 &= -\int (j \cdot \nabla) b_1 \, \omega_1 \, dx \, dy = \int j_1 \partial_x b_1 \omega_1 + j_2 \partial_y b_1 \omega_1 \\ &\leq \|b_1\|_\infty \left[\|\partial_x j_1\|_2 \|\omega_1\|_2 + \|j_1\| \|\partial_x \omega_1\|_2 \right] + \|\partial_y b_1\|_2 \|j_2\|_2^{\frac{1}{2}} \|\partial_y j_2\|_2^{\frac{1}{2}} \|\omega_1\|_2^{\frac{1}{2}} \|\partial_1 \omega_1\|_2^{\frac{1}{2}} \\ &\leq \frac{1}{16} \|\partial_x j_1\|_2^2 + \frac{1}{16} \|\partial_x \omega_1\|_2^2 + C \left(\|b_1\|_\infty^2 + \|\partial_y b_1\|_2^2 \right) Z(t) \end{aligned}$$

Similarly,

$$\begin{aligned} K_3 &\leq \| \left(\partial_x \omega_1, \partial_x \omega_2 \|_2^2 + C \left(\|\partial_x u_2\|_2^2 + \|u_1\|_\infty^2 \right) Z(t) \\ K_4 &\leq \frac{1}{16} \|(\partial_x j_1, \partial_x j_2)\|_2^2 + \|\partial_x \omega_1, \partial_x \omega_2\|_2^2 + \left(\|\partial_x b_2\|_2^2 + \|b_1\|_\infty^2 \right) Z(t) \\ K_5 &\leq \frac{1}{16} \|\partial_x j_1\|_2^2 + \|u_1\|_\infty^2 \|j_1\|_2^2 + \|\partial_y b_1\|_2^2 Z(t) \\ K_6 &\leq \frac{1}{16} \|\partial_x \omega_1\|_2^2 + \frac{1}{16} \|\partial_x j_1\|_2^2 + \left(\|b_1\|_\infty^2 + \|\partial_y b_1\|_2^2 \right) Z(t) \\ K_7 &\leq \frac{1}{16} \|(\partial_x j_1, \partial_x j_2)\|_2^2 + C \left(\|\partial_x u_2\|_2^2 + \|u_1\|_\infty^2 \right) Z(t) \\ K_8 &\leq \frac{1}{16} \left(\|\partial_x \omega_1, \partial_x \omega_2\|_2^2 + \frac{1}{16} \|(\partial_x j_1, \partial_x j_2)\|_2^2 + \left(\|\partial_x b_2\|_2^2 + \|b_1\|_\infty^2 \right) Z(t) \end{aligned}$$

After combining all inequalities

$$\frac{1}{2} \frac{d}{dt} Z(t) + \|(\partial_x \omega_1, \partial_x \omega_2)\|_2^2 + \|(\partial_x j_1, \partial_x j_2)\|_2^2 \leq \frac{1}{2} \|(\partial_x \omega_1, \partial_x \omega_2)\|_2^2 + \frac{1}{2} \|(\partial_x j_1, \partial_x j_2)\|_2^2 + CB(t) Z(t)$$

where $B(t)$ is integrable.

After applying Gronwal's lemma, we obtain the desired result.

3. H^2 bound

In this section, we establish the global H^2 bound for (u, b). For the notational convenience, we write ∇u for $\tilde{\nabla} u$. For instance, $\nabla \omega_3 = (\partial_x \omega_3, \partial_y \omega_3, \partial_z \omega_3) = (\partial_x \omega_3, \partial_y \omega_3, 0) = \tilde{\nabla} \omega_3$.

3.1. *Global bound for* $(\nabla \omega_3, \nabla j_3)$*:*

We consider the case $\nu_1 > 0$, $\nu_2 = 0$ $\eta_1 > 0$, $\eta_2 = 0$ in (1.3). Taking an inner product of the third and sixth equations in (1.3) with $(\Delta \omega_3, \Delta j_3)$, integrating in space to obtain

$$\frac{1}{2} \frac{d}{dt} \left(\|\nabla \omega_3\|_2^2 + \|\nabla j_3\|_2^2 \right) + \|\nabla \partial_x \omega_3\|_2^2 + \|\nabla \partial_x j_3\|_2^2 = L_1 + L_2 + L_3 + L_4 + L_5$$

where

$$\begin{aligned} L_1 &= -\int \nabla \omega_3 \cdot \nabla u \cdot \nabla \omega_3 \, dxdy, \quad L_2 = -\int \nabla j_3 \cdot \nabla u \cdot \nabla j_3 \, dxdy, \\ L_3 &= 2 \int \nabla \omega_3 \cdot \nabla b \cdot \nabla j_3 \, dxdy, \quad L_4 = 2 \int \nabla [\partial_x b_1 (\partial_x u_2 + \partial_y u_1)] \cdot \nabla j_3 \, dxdy, \\ L_5 &= -2 \int \nabla [\partial_x u_1 (\partial_x b_2 + \partial_y b_1)] \cdot \nabla j_3 \, dxdy. \end{aligned}$$

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$$
\begin{aligned}
L_1 &= \int (\nabla\omega_3 \cdot \nabla u \cdot \nabla\omega_3)\, dxdy \\
&= \int \left[\partial_x u_1 (\partial_x\omega_3)^2 + \partial_x u_2 \partial_x\omega_3 \partial_y\omega_3 + \partial_y u_1 \partial_x\omega_3\partial_y\omega_3 + \partial_y u_2 (\partial_y\omega_3)^2 \right) \, dxdy \big] \\
&= L_{11} + L_{12} + L_{13} + L_{14}
\end{aligned}
$$

$$
\begin{aligned}
L_{11} &\le C\|\partial_x u_1\|_2 \|\partial_x\omega_3\|_2^{\frac12} \|\partial_{xx}\omega_3\|_2^{\frac12} \|\partial_x\omega_3\|_2^{\frac12} \|\partial_{xy}\omega_3\|_2^{\frac12} \\
&\le \frac{1}{16}\|\nabla\partial_x\omega_3\|_2^2 + C\|\partial_x u_1\|_2^2 \|\nabla\omega_3\|_2^2 \\
L_{12} &\le C\|\partial_x u_2\|_2 \|\partial_x\omega_3\|_2^{\frac12} \|\partial_{xy}\omega_3\|_2^{\frac12} \|\partial_y\omega_3\|_2^{\frac12} \|\partial_{xy}\omega_3\|_2^{\frac12} \\
&\le C\|\partial_x u_2\|_2 \|\nabla\omega_3\|_2 \|\nabla\partial_x\omega_3\|_2 \\
L_{13} &\le C\|\partial_y u_1\|_2 \|\partial_y\omega_3\|_2^{\frac12} \|\partial xy\omega_3\|_2^{\frac12} \|\partial_x\omega_3\|_2^{\frac12} \|\partial_{xy}\omega_3\|_2^{\frac12} \\
&\le \|\partial_y u_1\|_2 \|\nabla\omega_3\|_2 \|\nabla\partial_x\omega_3\|_2 \\
L_{14} &\le C\|\partial_y\omega_3\|_2 \|\partial_x u_1\|_2^{\frac12} \|\partial_{xy} u_1\|_2^{\frac12} \|\partial_y\omega_3\|_2^{\frac12} \|\partial xy\omega_3\|_2^{\frac12} \\
&\le C\|\nabla\omega_3\|_2^{\frac32} \|\partial_x u_1\|_2^{\frac12} \|\partial_x\omega_3\|_2^{\frac12} \|\nabla\partial_x\omega_3\|_2^{\frac12} \\
&\le \frac{1}{16}\|\nabla\partial_x\omega_3\|_2^2 + C\|\nabla\omega_3\|_2^2 \|\partial_x u_1\|_2^{\frac23} \|\partial_x\omega_3\|_2^{\frac23}
\end{aligned}
$$

Thus,

$$
|L_1| \le \frac18 \|\nabla\partial_x\omega_3\|_2^2 + C\left(\|(\partial_x u_1, \partial_x u_2)\|_2^2 + \|\partial_y u_1\|_2^2\right) \|\nabla\omega_3\|_2^2 + C\|\nabla\omega_3\|_2^2 \|\omega_3\|_2^{\frac23} \|\partial_x\omega_3\|_2^{\frac23}
$$

Similarly, L_2, L_3, L_4, and, L_5 are bounded by

$$
\begin{aligned}
|L_2| &\le \frac18 \|\nabla\partial_x j_3\|_2^2 + C\left(\|\omega_3\|_2^2 + \|\omega_3\|_2^{\frac23} \|\partial_x\omega_3\|_2^{\frac23}\right) \|\nabla j_3\|_2^2 \\
|L_3| &\le \frac18 \|\nabla\partial_x\omega_3\|_2^2 + \frac18 \|\nabla\partial_x j_3\|_2^2 + C\, \|j_3\|_2^2 \left(\|\nabla\omega_3\|_2^2 + \|\nabla j_3\|_2^2\right) + C\, \|\partial_x j_3\|_2^2 \|\nabla j_3\|_2^2, \\
|L_4| &\le \frac18 \|\nabla\partial_x\omega_3\|_2^2 + \frac18 \|\nabla\partial_x j_3\|_2^2 \\
&\quad + C\left(\|j_3\|_2^2 + \|\omega_3\|_2^2 + \|\partial_x j_3\|_2^2 + \|\omega_3\|_2^{\frac23} \|\partial_x\omega_3\|_2^{\frac23}\right)\left(\|\nabla\omega_3\|_2^2 + \|\nabla j_3\|_2^2\right), \\
|L_5| &\le \frac18 \|\nabla\partial_x\omega_3\|_2^2 + \frac18 \|\nabla\partial_x j_3\|_2^2 \\
&\quad + C\left(\|j_3\|_2^2 + \|\omega_3\|_2^2 + \|\partial_x j_3\|_2^2 + \|\partial_x\omega_3\|_2^2\right)\left(\|\nabla\omega_3\|_2^2 + \|\nabla j_3\|_2^2\right).
\end{aligned}
$$

Combining these estimates, applying Gronwall's inequality and invoking the global H^1 bound, we prove global bound for $\|(\nabla\omega_3, \nabla j_3)\|_2^2$.

3.2. *Global bound for* $(\nabla\omega_1, \nabla\omega_2)$ *and* $(\nabla j_1, \nabla j_2)$

We consider the case $\nu_1 > 0$, $\nu_2 = 0$ $\eta_1 > 0$, $\eta_2 = 0$ in (1.3). Taking an inner product of the first, second, third, and fifth equations in (1.3) with $(\Delta\omega_1, \Delta\omega_2)$ and $(\Delta j_1, \Delta j_2)$, integrating in space to obtain

$$
\begin{aligned}
&\frac12 \frac{d}{dt}\left[\|(\nabla\omega_1, \nabla\omega_2)\|_2^2 + \|(\nabla j_1, \nabla j_2)\|_2^2\right] + \|(\nabla\partial_x\omega_1, \nabla\partial_x\omega_2)\|_2^2 + \|(\nabla\partial_x j_1, \nabla\partial_x j_2)\|_2^2 \\
&\quad = \sum I_i
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \int (\nabla\omega_1 \cdot \nabla u \cdot \nabla\omega_1)\, dxdy, \qquad I_2 = \int ((\omega\cdot\nabla)u_1 - (j\cdot\nabla b_1)\cdot \Delta\omega_1 \, dx\, dy \\
I_3 &= \int ((\omega\cdot\nabla)u_2 - (j\cdot\nabla b_2)\cdot\Delta\omega_2\, dx\, dy, \qquad I_4 = \int (b\cdot\nabla)j_1\cdot\Delta\omega_1\, dx\, dy \\
I_5 &= \int (\nabla\omega_2\cdot\nabla u\cdot\nabla\omega_2)\, dx\, dy, \qquad I_6 = \int ((j\cdot\nabla)u_1 - (\omega\cdot\nabla b_1)\cdot\Delta j_1 \\
I_7 &= \int (b\cdot\nabla)j_1\cdot\Delta\omega_1\, dx\, dy \qquad I_8 = \int ((j\cdot\nabla)u_2 - (\omega\cdot\nabla b_2)\cdot\Delta j_2\, dx\, dy \\
I_9 &= \int (\nabla j_1\cdot\nabla u\cdot\nabla j_1)\, dx\, dy, \qquad I_{10} = \int (\nabla j_2\cdot\nabla u\cdot\nabla j_2)\, dxdy
\end{aligned}
$$

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For notational convenience, set $X(t) = \|(\nabla\omega_1, \nabla\omega_2)\|_2^2 + \|(\nabla j_1, \nabla j_2)\|_2^2$

$$\begin{aligned} I_1 &= \int (\nabla\omega_1 \cdot \nabla u \cdot \nabla\omega_1)\ dxdy \\ &= \int \left[\partial_x u_1 (\partial_x\omega_1)^2 + \partial_x u_2 \partial_x\omega_1 \partial_y\omega_1 + \partial_y u_1 \partial_x\omega_1 \partial_y\omega_1 + \partial_y u_2 (\partial_y\omega_1)^2 \right)\ dxdy \big] \\ &= I_{11} + I_{12} + I_{13} + I_{14} \end{aligned}$$

$$\begin{aligned} I_{11} &\le C\|\partial_x u_1\|_2 \|\partial_x\omega_1\|_2^{\frac12} \|\partial_{xx}\omega_1\|_2^{\frac12} \|\partial_x\omega_1\|_2^{\frac12} \|\partial_{xy}\omega_1\|_2^{\frac12} \\ &\le \frac{1}{16}\|\nabla\partial_x\omega_1\|_2^2 + C\|\partial_x u_1\|_2^2 \|\nabla\omega_1\|_2^2 \\ I_{12} &\le C\|\partial_x u_2\|_2 \|\partial_x\omega_1\|_2^{\frac12} \|\partial_{xy}\omega_1\|_2^{\frac12} \|\partial_y\omega_1\|_2^{\frac12} \|\partial_{xy}\omega_1\|_2^{\frac12} \\ &\le C\|\partial_x u_2\|_2 \|\nabla\omega_1\|_2 \|\nabla\partial_x\omega_1\|_2 \\ I_{13} &\le C\|\partial_y u_1\|_2 \|\partial_y\omega_1\|_2^{\frac12} \|\partial xy\omega_1\|_2^{\frac12} \|\partial_x\omega_1\|_2^{\frac12} \|\partial_{xy}\omega_1\|_2^{\frac12} \\ &\le \|\partial_y u_1\|_2 \|\nabla\omega_1\|_2 \|\nabla\partial_x\omega_1\|_2 \\ I_{14} &\le C\|\partial_y\omega_1\|_2 \|\partial_x u_1\|_2^{\frac12} \|\partial_{xy} u_1\|_2^{\frac12} \|\partial_y\omega_1\|_2^{\frac12} \|\partial xy\omega_1\|_2^{\frac12} \\ &\le C\|\nabla\omega_1\|_2^{\frac32} \|\partial_x u_1\|_2^{\frac12} \|\partial_x\omega_1\|_2^{\frac12} \|\nabla\partial_x\omega_1\|_2^{\frac12} \\ &\le C\frac{1}{16}\|\nabla\partial_x\omega_1\|_2^2 + C\|\nabla\omega_1\|_2^2 \|\partial_x u_1\|_2^{\frac23} \|\partial_x\omega_1\|_2^{\frac23} \end{aligned}$$

Thus,

$$|I_1| \le \frac18\|\nabla\partial_x\omega_1\|_2^2 + C\left(\|(\partial_x u_1, \partial_x u_2)\|_2^2 + \|\partial_y u_1\|_2^2\right)\|\nabla\omega_1\|_2^2 + C\|\nabla\omega_1\|_2^2 \|\omega_1\|_2^{\frac23} \|\partial_x\omega_1\|_2^{\frac23}$$

Similarly,

$$\begin{aligned} |I_5 &\le \frac18\|\nabla\partial_x\omega_2\|_2^2 + C\left(\|(\partial_x u_1, \partial_x u_2)\|_2^2 + \|\partial_y u_1\|_2^2\right)\|\nabla\omega_2\|_2^2 + C\|\nabla\omega_2\|_2^2 \|\omega_2\|_2^{\frac23} \|\partial_x\omega_2\|_2^{\frac23} \\ |I_9| &\le \frac18\|\nabla\partial_x j_1\|_2^2 + C\left(\|(\partial_x u_1, \partial_x u_2)\|_2^2 + \|\partial_y u_1\|_2^2\right)\|\nabla j_1\|_2^2 + C\|\nabla j_1\|_2^2 \|j_1\|_2^{\frac23} \|\partial_x j_1\|_2^{\frac23} \\ |I_{10}| &\le \frac18\|\nabla\partial_x j_2\|_2^2 + C\left(\|(\partial_x u_1, \partial_x u_2)\|_2^2 + \|\partial_y u_1\|_2^2\right)\|\nabla j_2\|_2^2 + C\|\nabla j_2\|_2^2 \|j_2\|_2^{\frac23} \|\partial_x j_2\|_2^{\frac23} \end{aligned}$$

Other terms can be bounded (similar to [18]) by

$$\begin{aligned} I_2 &\le \frac{1}{16}\|\partial_{xy}\omega_1\|_2^2 + C\|\nabla\omega_1\|_2^2 + C\left[\|(\|\omega, j)\|_2^2 + \|(\nabla\omega_3, \nabla j_3)\|_2^2\right] X(t) \\ I_3 &\le \frac{1}{16}\|\partial_{xy}\omega_2\|_2^2 + C\|\nabla\omega_2\|_2^2 + C\left[\|(\|\omega, j)\|_2^2 + \|(\nabla\omega_3, \nabla j_3)\|_2^2\right] X(t) \\ I_4 + I_7 &\le \frac{1}{16}\|(\partial_{xy} j_1, \partial_{xy} j_2)\|_2^2 + C\|(j_3, \nabla j_3)\|_2^2 X(t) \\ I_6 &\le \frac{1}{16}\|\partial_{xy} j_1\|_2^2 + C\|\nabla j_1\|_2^2 + C\left(\|\omega, j\|_2^2 + C\|\nabla\omega_3, \nabla j_3\|_2^2\right) X(t) \\ I_8 &\le \frac{1}{16}\|\partial_{xy} j_2\|_2^2 + C\|\nabla j_2\|_2^2 + C\left(\|\omega, j\|_2^2 + C\|\nabla\omega_3, \nabla j_3\|_2^2\right) X(t) \end{aligned}$$

After combing all inequalities and together with Gronwall's , we obtain the H^1-bound for $\omega_1, \omega_2, j_1, j_2$. Therefore, the global H^1 bound for $\omega_1, \omega_2, \omega_3$ together with the global H^1 bound for j_1, j_2, j_3, we obtain the global H^2 bound for (u, b) for two-and-half-dimensional MHD equations with horizontal dissipation and horizontal magnetic diffusion.

Theorem 1.2 can be proved similarly as theorem 1.1.

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