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Global Regularity for the 2D Magneto-Micropolar Equations with Partial Dissipation

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Abstract. This paper studies the global existence and regularity of classical solutions to the 2D incompressible magneto-micropolar equations with partial dissipation. The magneto-micropolar equations model the motion of electrically conducting micropolar fluids in the presence of a magnetic field. When there is only partial dissipation, the global regularity problem can be quite difficult. We are able to single out three special partial dissipation cases and establish the global regularity for each case. As special consequences, the 2D Navier-Stokes equations, the 2D magnetohydrodynamic equations, and the 2D micropolar equations with several types of partial dissipation always possess global classical solutions. The proofs of our main results rely on anisotropic Sobolev type inequalities and suitable combination and cancellation of terms.

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Key words: Global regularity, magneto-micropolar equations, partial dissipation.

1 Introduction

This paper aims at the global existence and regularity of classical solutions to the 2D incompressible magneto-micropolar equations with partial dissipation. The standard 3D incompressible magneto-micropolar equations can be written as

$$\begin{cases} \partial_t u+(u\cdot\nabla)u+\nabla(p+\frac{1}{2}|b|^2)=(\mu+\chi)\Delta u+(b\cdot\nabla)b+2\chi\nabla\times\omega, \\ \partial_t b+(u\cdot\nabla)b=\nu\Delta b+(b\cdot\nabla)u, \\ \partial_t\omega+(u\cdot\nabla)\omega+2\chi\omega=\kappa\Delta\omega+(\alpha+\beta)\nabla\nabla\cdot\omega+2\chi\nabla\times u, \\ \nabla\cdot u=0, \quad \nabla\cdot b=0, \end{cases} \tag{1.1}$$

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where, for $\mathbf{x}\in\mathbb{R}^3$ and $t\geq 0$, $u=u(\mathbf{x},t),b=b(\mathbf{x},t),\omega=\omega(\mathbf{x},t)$ and $p=p(\mathbf{x},t)$ denote the velocity field, the magnetic field, the micro-rotation field and the pressure, respectively, and μ denotes the kinematic viscosity, ν the magnetic diffusivity, χ the vortex viscosity, and α, and β and κ the angular viscosities. The 3D magneto-micropolar equations reduce to the 2D magneto-micropolar equations when

$$\begin{aligned}&u=(u_1(x,y,t),u_2(x,y,t),0), \quad b=(b_1(x,y,t),b_2(x,y,t),0),\\&\omega=(0,0,\omega(x,y,t)), \quad \pi=\pi(x,y,t),\end{aligned}$$

where $(x,y)\in\mathbb{R}^2$ and we have written $\pi=p+\frac{1}{2}|b|^2$. More explicitly, the 2D magneto-micropolar equations can be written as

$$\begin{cases}\partial_t u+(u\cdot\nabla)u+\nabla\pi=(\mu+\chi)\Delta u+(b\cdot\nabla)b+2\chi\nabla\times\omega,\\ \partial_t b+(u\cdot\nabla)b=\nu\Delta b+(b\cdot\nabla)u,\\ \partial_t\omega+(u\cdot\nabla)\omega+2\chi\omega=\kappa\Delta\omega+2\chi\nabla\times u,\\ \nabla\cdot u=0, \quad \nabla\cdot b=0,\end{cases} \tag{1.2}$$

where $u=(u_1,u_2)$, $b=(b_1,b_2)$, $\nabla\times\omega=(-\partial_y\omega,\partial_x\omega)$ and $\nabla\times u=\partial_x u_2-\partial_y u_1$.

The magneto-micropolar equations model the motion of electrically conducting micropolar fluids in the presence of a magnetic field. Micropolar fluids represent a class of fluids with nonsymmetric stress tensor (called polar fluids) such as fluids consisting of suspending particles, dumbbell molecules, etc (see, e.g., [6,8–10,17]). A generalization of the 2D magneto-micropolar equations is given by

$$\begin{cases}\partial_t u_1+(u\cdot\nabla)u_1+\partial_x\pi=\mu_{11}\partial_{xx}u_1+\mu_{12}\partial_{yy}u_1+(b\cdot\nabla)b_1-2\chi\partial_y\omega,\\ \partial_t u_2+(u\cdot\nabla)u_2+\partial_y\pi=\mu_{21}\partial_{xx}u_2+\mu_{22}\partial_{yy}u_2+(b\cdot\nabla)b_2+2\chi\partial_x\omega,\\ \partial_t b+(u\cdot\nabla)b=\nu_1\partial_{xx}b+\nu_2\partial_{yy}b+(b\cdot\nabla)u,\\ \partial_t\omega+(u\cdot\nabla)\omega+2\chi\omega=\kappa_1\partial_{xx}\omega+\kappa_2\partial_{yy}\omega+2\chi\nabla\times u,\\ \nabla\cdot u=0, \quad \nabla\cdot b=0,\\ u(x,y,0)=u_0(x,y),b(x,y,0)=b_0(x,y),\omega(x,y,0)=\omega_0(x,y),\end{cases} \tag{1.3}$$

where we have written the velocity equation in its two components. Clearly, if

$$\mu_{11}=\mu_{12}=\mu_{21}=\mu_{22}=\mu+\chi, \quad \nu_1=\nu_2=\nu, \quad \kappa_1=\kappa_2=\kappa,$$

then (1.3) reduces to the standard 2D magneto-micropolar equations in (1.2). This generalization is capable of modeling the motion of anisotropic fluids for which the diffusion properties in different directions are different. In addition, (1.3) allows us to explore the smoothing effects of various partial dissipations.

The magneto-micropolar equations above are not only important in engineering and physics, but also mathematically significant. The mathematical study of the magneto-micropolar equations started in the seventies and has been continued by many authors (see, e.g., [1,18,21,29,30]). Some of the recent efforts are devoted to the well-posedness problem and various asymptotic behavior. The focus of this paper will be on the global existence and uniqueness problem on the generalized 2D magneto-micropolar equations (1.3) with various partial dissipation. We deal with three main partial dissipation cases and establish the global regularity for each case. For notational convenience, we set $\chi = 1/2$ for the rest of the paper.

The first partial dissipation case corresponds to (1.3) with

$$\mu_{11}=\mu_{22}=0,\quad \nu_2=0,\quad \kappa_2=0,\quad \mu_{12}=\mu_{21}=1,\quad \nu_1=\kappa_1=1,$$

or, more precisely,

$$\begin{cases} \partial_t u_1+(u\cdot\nabla)u_1+\partial_x\pi=\partial_{yy}u_1+(b\cdot\nabla)b_1-\partial_y\omega, \\ \partial_t u_2+(u\cdot\nabla)u_2+\partial_y\pi=\partial_{xx}u_2+(b\cdot\nabla)b_2+\partial_x\omega, \\ \partial_t b+(u\cdot\nabla)b=\partial_{xx}b+b\cdot\nabla u, \\ \partial_t\omega+(u\cdot\nabla)\omega+\omega=\partial_{xx}\omega+\nabla\times u, \\ \nabla\cdot u=0,\quad \nabla\cdot b=0, \\ u(x,y,0)=u_0(x,y), b(x,y,0)=b_0(x,y), \omega(x,y,0)=\omega_0(x,y). \end{cases} \tag{1.4}$$

The global existence and regularity result for this case can be stated as follows.

Theorem 1.1. *Assume $(u_0,b_0,\omega_0)\in H^2(\mathbb{R}^2)$, and $\nabla\cdot u_0=\nabla\cdot b_0=0$. Then* (1.4) *has a unique global classical solution (u,b,ω) satisfying, for any $T>0$,*

$$(u,b,\omega)\in L^\infty([0,T];H^2(\mathbb{R}^2)).$$

The second partial dissipation case corresponds to (1.3) with

$$\mu_{11}=1,\quad \mu_{22}=\nu_2=\kappa_2=0,\quad \mu_{12}=1,\quad \mu_{21}=0,\quad \nu_1=\kappa_1=1,$$

or, more precisely,

$$\begin{cases} \partial_t u_1+(u\cdot\nabla)u_1+\partial_x\pi=\Delta u_1+(b\cdot\nabla)b_1-\partial_y\omega, \\ \partial_t u_2+(u\cdot\nabla)u_2+\partial_y\pi=(b\cdot\nabla)b_2+\partial_x\omega, \\ \partial_t b+(u\cdot\nabla)b=\partial_{xx}b+(b\cdot\nabla)u, \\ \partial_t\omega+(u\cdot\nabla)\omega+\omega=\partial_{xx}\omega+\nabla\times u, \\ \nabla\cdot u=0,\quad \nabla\cdot b=0, \\ u(x,y,0)=u_0(x,y), b(x,y,0)=b_0(x,y), \omega(x,y,0)=\omega_0(x,y). \end{cases} \tag{1.5}$$

The global well-posedness for (1.5) is given in the following theorem.

Theorem 1.2. *Assume* $(u_0,b_0,\omega_0)\in H^2(\mathbb{R}^2)$*, and* $\nabla\cdot u_0=\nabla\cdot b_0=0$*. Then* (1.5) *has a unique global classical solution* (u,b,ω) *satisfying, for any* $T>0$*,*

$$(u,b,\omega)\in L^\infty([0,T];H^2(\mathbb{R}^2)).$$

Our third main result establishes the global regularity for the partial dissipation case (1.3) with

$$\mu_{12}=\mu_{22}=1,\quad \mu_{11}=\mu_{21}=\nu_2=\kappa_2=0,\quad \nu_1=\kappa_1=1,$$

or, more precisely,

$$\begin{cases} \partial_t u_1+(u\cdot\nabla)u_1+\partial_x\pi=\partial_{yy}u_1+(b\cdot\nabla)b_1-\partial_y\omega,\\ \partial_t u_2+(u\cdot\nabla)u_2+\partial_y\pi=\partial_{yy}u_2+(b\cdot\nabla)b_2+\partial_x\omega,\\ \partial_t b+(u\cdot\nabla)b=\partial_{xx}b+(b\cdot\nabla)u,\\ \partial_t\omega+(u\cdot\nabla)\omega+\omega=\partial_{xx}\omega+\nabla\times u,\\ \nabla\cdot u=0,\quad \nabla\cdot b=0,\\ u(x,y,0)=u_0(x,y),b(x,y,0)=b_0(x,y),\omega(x,y,0)=\omega_0(x,y). \end{cases} \tag{1.6}$$

Theorem 1.3. *Assume* $(u_0,b_0,\omega_0)\in H^2(\mathbb{R}^2)$*, and* $\nabla\cdot u_0=\nabla\cdot b_0=0$*. Then* (1.6) *has a unique global classical solution* (u,b,ω) *satisfying, for any* $T>0$*,*

$$(u,b,\omega)\in L^\infty([0,T];H^2(\mathbb{R}^2)).$$

It is worth mentioning some of the special consequences of our theorems. In the special case when $b\equiv 0$ and $\omega\equiv 0$, the magneto-micropolar equations become the 2D Navier-Stokes equations and the theorems above assess the global regularity for the Navier-Stokes with various partial dissipation. These results for the Navier-Stokes equations appear to be new.

Corollary 1.1. *Consider* (1.4), (1.5) *or* (1.6) *with* $b\equiv 0$ *and* $\omega\equiv 0$*. Assume* $u_0\in H^2(\mathbb{R}^2)$*, and* $\nabla\cdot u_0=0$*. Then any one of these systems has a unique global solution.*

When $\omega\equiv 0$, the magneto-micropolar equations become the magneto-hydrodynamic (MHD) equations. The results in the first two theorems are new for the MHD equations while the third one recovers a result in [3].

Corollary 1.2. *Consider* (1.4), (1.5) *or* (1.6) *with* $\omega\equiv 0$*. Assume* $(u_0,b_0)\in H^2(\mathbb{R}^2)$*, and* $\nabla\cdot u_0=\nabla\cdot b_0=0$*. Then any one of these systems has a unique global solution.*

When $b\equiv 0$, the magneto-micropolar equations become the micropolar equations and the results in the theorems above reduce to those for the micropolar equations.

Corollary 1.3. *Consider* (1.4), (1.5) *or* (1.6) *with* $b\equiv 0$*. Assume* $(u_0,\omega_0)\in H^2(\mathbb{R}^2)$ *and* $\nabla\cdot u_0=0$*. Then any one of these systems has a unique global solution.*

Now we explain the main difficulties involved in proving the theorems and the methods used here. The general approach to establish the global existence and regularity results consists of two main steps. The first step assesses the local (in time) well-posedness while the second extends the local solution into a global one by obtaining global (in time) *a priori* bounds. For the systems of equations concerned here, the local well-posedness follows from a standard approach and shall be skipped here. Our main efforts are devoted to proving the necessary global *a priori* bounds. More precisely, we show that, for any $T>0$ and $t\le T$,

$$\|(u,b,\omega)(\cdot,t)\|_{H^2(\mathbb{R}^2)}\le C, \tag{1.7}$$

where C denotes a bound that depends on T and the initial data. In general, we rely on the smoothing effects of the dissipative terms in the systems. When there is no dissipation in (1.3), it is impossible to prove (1.7). Then the issue is how much dissipation we really need in order to prove (1.7). We are able to single out the aforementioned three partial dissipation cases and prove (1.7).

The proof of (1.7) involves three steps. The first step proves the global L^2-bound. This step is easy and relies on the divergence-free condition $\nabla\cdot u=\nabla\cdot b=0$. The second step proves the global H^1-bound for (u,b,ω). This step is not trivial and fully exploits the partial dissipation. This step also makes use of the anisotropic Sobolev type inequalities (see Lemma 2.1). This last step is to prove the global H^2-bound by using the global H^1-bound and various anisotropic inequalities. The whole process involves the estimates of many terms and is complex. The details are given in the subsequent sections.

We briefly mention some of closely related results. In [29] Yamazaki obtained the global regularity of the 2D magneto-micropolar equation with zero angular viscosity, namely (1.2) with $\kappa=0$ and other coefficients being positive. Another partial dissipation case for the 2D magneto-micropolar equation was studied in [5]. As aforementioned, quite a few global regularity results for the 2D MHD and the 2D micropolar equations with partial dissipation are available (see, e.g., [2–4,7,11–16,19,20,22–28,31]).

The rest of this paper is divided into three sections with each of them devoted to the proof of one of the theorems stated above. To simplify the notation, we will write $\|f\|_2$ for $\|f\|_{L^2}$, $\int f$ for $\int_{\mathbb{R}^2} f\,dxdy$ and write $\frac{\partial}{\partial x}f$, $\partial_x f$ or f_x as the first partial derivative, and $\frac{\partial^2}{\partial x^2}f$ or $\partial_{xx}f$ as the second partial throughout the rest of this paper.

2 Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. As explained in the introduction, it suffices to establish the global *a priori* bound for the solution in H^2. For the sake of clarity, we divide this process into two subsections. The first subsection proves the global H^1-bound while the second proves the global H^2-bound.

In the proof of Theorem 1.1, the following anisotropic type Sobolev inequality will be frequently used. Its proof can be found in [3].

Lemma 2.1. *If $f,g,h,\partial_y g,\partial_x h\in L^2(\mathbb{R}^2)$, then*

$$\iint_{\mathbb{R}^2} |fgh|\,dxdy \le C\|f\|_2\,\|g\|_2^{\frac{1}{2}}\,\|\partial_y g\|_2^{\frac{1}{2}}\,\|h\|_2^{\frac{1}{2}}\,\|\partial_x h\|_2^{\frac{1}{2}}, \tag{2.1}$$

where C is a constant.

The following simple fact on the boundedness of Riesz transforms will also be used.

Lemma 2.2. *Let f be divergence-free vector field such that $\nabla f\in L^p$ for $p\in(1,\infty)$. Then there exists a pure constant $C>0$ (independent of p) such that*

$$\|\nabla f\|_{L^p} \le \frac{Cp^2}{p-1}\|\nabla\times f\|_{L^p}.$$

The rest of this section is divided into two subsections. The first subsection proves the global H^1-bound while the second proves the global H^2-bound.

2.1 H^1-Bound

We first state the global L^2-bound.

Lemma 2.3. *Assume that (u_0,b_0,ω_0) satisfies the condition stated in Theorem* 1.1. *Let (u,b,ω) be the corresponding solution of* (1.4). *Then, (u,b,ω) obeys the following global L^2-bound,*

$$\begin{aligned}\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+2\int_0^t\|(\partial_y u_1,\partial_x u_2)\|_{L^2}^2 d\tau\\ +2\int_0^t\|\partial_x b(\tau)\|_{L^2}^2 d\tau+2\int_0^t\|\partial_x\omega(\tau)\|_{L^2}^2 d\tau\le C(\|(u_0,b_0,\omega_0)\|_2^2)\end{aligned}$$

for any $t\ge 0$.

Proof. The proof of the global L^2-bound is easy. Taking the L^2-inner product of (u,b,ω) with (1.4), respectively, yields

$$\begin{aligned}&\frac{1}{2}\partial_t\|u\|_2^2+\|(\partial_y u_1,\partial_x u_2)\|_2^2=\int(b\cdot\nabla)b\cdot u+\int(\nabla^\perp\omega)\cdot u,\\ &\frac{1}{2}\partial_t\|b\|_2^2+\|\partial_x b\|_2^2=\int(b\cdot\nabla)u\cdot b,\\ &\frac{1}{2}\partial_t\|\omega\|_2^2+\|\omega_x\|_2^2+\|\omega\|_2^2=\int(\nabla\times u)\omega.\end{aligned}$$

Adding them up and using the fact

$$\int(b\cdot\nabla)b\cdot u+\int(b\cdot\nabla)u\cdot b=0,\qquad \int\nabla^\perp\omega\cdot u=\int(\nabla\times u)\omega,$$

we have

$$\frac{1}{2}\partial_t(\|u\|_2^2+\|b\|_2^2+\|\omega\|_2^2)+\|(\partial_y u_1,\partial_x u_2)\|_2^2+\|\partial_x b\|_2^2+\|\omega_x\|_2^2+\|\omega\|_2^2=2\int(\nabla\times u)\omega.$$

To bound the right-hand side, we notice that

$$2\int(\nabla\times u)\omega=2\int(\partial_x u_2-\partial_y u_1)\omega=-2\int\partial_y u_1\omega dxdy-2\int\partial_x\omega u_2.$$

Applying Hölder's inequality yields

$$\begin{aligned}&\frac{1}{2}\partial_t(\|u\|_2^2+\|b\|_2^2+\|\omega\|_2^2)+\|(\partial_y u_1,\partial_x u_2)\|_2^2+\|\partial_x b\|_2^2+\|\omega_x\|_2^2+\|\omega\|_2^2\\ \le&\frac{1}{2}(\|\partial_y u_1\|_2^2+\|\partial_x\omega\|_2^2)+C(\|u\|_2^2+\|\omega\|_2^2).\end{aligned}$$

Gronwall's inequality then implies

$$\|u\|_2^2+\|b\|_2^2+\|\omega\|_2^2+\int_0^t\left(\|(\partial_y u_1,\partial_x u_2)\|_2^2+\|\partial_x b\|_2^2+\|\omega_x\|_2^2+\|\omega\|_2^2\right)d\tau\le C,$$

for any $0<t\le T$, where C depends only on the initial data. ☐

We next prove the global H^1-bound for u,b and ω.

Proposition 2.1. Assume that (u_0,b_0,ω_0) satisfies the condition stated in Theorem 1.1. Let (u,b,ω) be the corresponding solution of (1.4). Then (u,b,ω) satisfies, for any $T>0$,

$$u,b,\omega\in C([0,T];H^1). \tag{2.2}$$

Proof of Proposition 2.1. To estimate the H^1-norm of (u,b,ω), we consider the equations of $\Omega=\nabla\times u$, $\nabla\omega$ and of the current density $j=\nabla\times b$,

$$\begin{aligned}&\Omega_t+u\cdot\nabla\Omega=\partial_{xxx}u_2-\partial_{yyy}u_1+(b\cdot\nabla)j-\Delta\omega, &(2.3)\\ &\partial_t\nabla\omega+\nabla(u\cdot\nabla\omega)+2\nabla\omega=\nabla\omega_{xx}+\nabla\Omega, &(2.4)\\ &j_t+u\cdot\nabla j=\partial_{xx}j+b\cdot\nabla\Omega+2\partial_x b_1(\partial_x u_2+\partial_y u_1)-2\partial_x u_1(\partial_x b_2+\partial_y b_1), &(2.5)\\ &\nabla\cdot u=0,\quad \nabla\cdot b=0.\end{aligned}$$

Dotting (2.3) by Ω, (2.4) by $\nabla\omega$ and (2.5) by j, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\Omega\|_{L^2}^2+\|j\|_{L^2}^2+\|\nabla\omega\|_2^2)+\|\nabla\partial_y u_1\|_2^2\\ &\qquad+\|\nabla\partial_x u_2\|_2^2+\|\partial_x j\|_{L^2}^2+\|\nabla\omega_x\|_2^2+2\|\nabla\omega\|_2^2\\ &=2\int\left[\partial_x b_1(\partial_x u_2+\partial_y u_1)j-\partial_x u_1(\partial_x b_2+\partial_y b_1)\right]jdxdy\end{aligned}$$

$$
\begin{aligned}
&-\int \nabla\omega\cdot\nabla u\cdot\nabla\omega+2\int\nabla\omega\cdot\nabla\Omega\\
\equiv\ & J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}
$$

Invoking the divergence-free condition, we note that

$$
\begin{aligned}
\|\nabla\partial_x u_1\|_2^2&=\|\partial_{xx}u_1\|_2^2+\|\partial_{yy}u_2\|_2^2,\\
\|\nabla\partial_y u_1\|_2^2&=\|\partial_{yy}u_2\|_2^2+\|\partial_{yy}u_1\|_2^2,\\
\|\nabla\partial_x u_2\|_2^2&=\|\partial_{xx}u_1\|_2^2+\|\partial_{xx}u_2\|_2^2.
\end{aligned}
$$

We now estimate the terms on the right. Since $j=\partial_x b_2-\partial_y b_1$,

$$
\begin{aligned}
J_1&=2\int\partial_x b_1\partial_x u_2\partial_x b_2-2\int\partial_x b_1\partial_x u_2\partial_y b_1\\
&\equiv J_{11}+J_{12}.
\end{aligned}
$$

Applying lemma 2.1, Young's inequality, and the simple fact that

$$
\|\partial_x b_2\|_{L^2}\le\|j\|_{L^2},\qquad \|\partial_{xy}b_1\|_{L^2}\le\|\partial_x j\|_{L^2},
$$

we have

$$
\begin{aligned}
J_{11}&\le 2\left|\int\partial_x b_1\partial_x u_2\partial_x b_2\right|\\
&\le C\|\partial_x u_2\|_2\|\partial_x b_1\|_2^{\frac12}\|\partial_{xy}b_1\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_{xx}b_2\|_2^{\frac12}\\
&\le C\|\Omega\|_2\|\partial_x b_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\\
&\le C\|\Omega\|_2\|\partial_x b\|_2\|\partial_x j\|_2\\
&\le\frac{1}{48}\|\partial_x j\|_2^2+C\|\partial_x b\|_2^2\|\Omega\|_2^2.
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
J_{12}&=2\int\partial_x b_1\partial_x u_2\partial_y b_1=-2\int u_2\partial_{xx}b_1\partial_y b_1-2\int u_2\partial_x b_1\partial_{xy}b_1\\
&\equiv J_{121}+J_{122}.\\
J_{121}&\le\left|-2\int u_2\partial_{xx}b_1\partial_y b_1\right|\\
&\le C\|\partial_{xx}b_1\|\|u_2\|_2^{\frac12}\|\partial_y u_2\|_2^{\frac12}\|\partial_y b_1\|_2^{\frac12}\|\partial_{xy}b_1\|_2^{\frac12}\\
&\le C\|\partial_x j\|_2\|u_2\|_2^{\frac12}\|\Omega\|_2^{\frac12}\|j\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\\
&\le\frac{1}{48}\|\partial_x j\|_2^2+C\|u_2\|_2\|j\|_2\|\Omega\|_2.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
J_{122} &\le \left| -2\int u_2 \partial_x b_1 \partial_{xy} b_1 \right| \\
&\le \frac{1}{48}\|\partial_x j\|_2^2 + C\|u_2\|_2 \|j\|_2 \|\Omega\|_2.
\end{aligned}
$$

J_2, J_3, and J_4 can be bounded by

$$
\begin{aligned}
J_2 &\le \left| \int \partial_x b_1 \partial_y u_1 j \right| \\
&\le C\|\partial_x b_1\|_2^{\frac12} \|\partial_{xx} b_1\|_2^{\frac12} \|\partial_y u_1\|_2^{\frac12} \|\partial_{yy} u_1\|_2^{\frac12} \|j\|_2 \\
&\le \frac{1}{48}\left[\|\partial_{yy} u_1\|_2^2 + \|\partial_x j\|_2^2\right] + C(\|\partial_x b_1\|_2^2 + \|\partial_y u_1\|_2^2)\|j\|_2^2. \\
J_3 &\le \left| \int \partial_x u_1 \partial_x b_2 j \right| \le \int \left| (u_1 \partial_{xx} b_2 j + u_1 \partial_x b_2 \partial_x j) \right| \\
&\le C\|u_1\|_2^{\frac12} \|\partial_y u_1\|_2^{\frac12} \|j\|_2^{\frac12} \|\partial_x j\|_2^{\frac12} \|\partial_{xx} b_2\|_2 \\
&\quad + C\|u_1\|_2^{\frac12} \|\partial_y u_1\|_2^{\frac12} \|\partial_x b_2\|_2^{\frac12} \|\partial_{xx} b_2\|_2^{\frac12} \|\partial_x j\|_2 \\
&\le \frac{1}{48}\|\partial_x j\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|j\|_2^2. \\
J_4 &\le \left| \int \partial_x u_1 \partial_y b_1 j \right| \le \int \left| (u_1 \partial_{xy} b_1 j - u_1 \partial_y b_1 j_x) \right| \\
&\le C\|u_1\|_2^{\frac12} \|\partial_y u_1\|_2^{\frac12} \|j\|_2^{\frac12} \|\partial_x j\|_2^{\frac12} \|\partial_{xy} b_1\|_2 \\
&\quad + C\|u_1\|_2^{\frac12} \|\partial_y u_1\|_2^{\frac12} \|\partial_y b_1\|_2^{\frac12} \|\partial_{xy} b_1\|_2^{\frac12} \|j_x\|_2 \\
&\le \frac{1}{48}\|\partial_x j\|_2^2 + C\|u_1\|_2^2 \|\partial_y u_1\|_2^2 \|j\|_2^2.
\end{aligned}
$$

To bound J_5, we use $\nabla \cdot u = 0$ and integrate by parts to obtain

$$
\begin{aligned}
J_5 &= -\int \nabla\omega \cdot \nabla u \cdot \nabla\omega \\
&= -2\int u_1 \omega_{xx} \omega_x - 2\int u_1 \omega_y \omega_{xy} - \int (\partial_x u_2 + \partial_y u_1)\omega_x \omega_y.
\end{aligned}
$$

The terms on the right can be bounded as

$$
\begin{aligned}
\left| \int u_1 \omega_{xx} \omega_x \right| &\le C\|\omega_{xx}\|_2 \|u_1\|_2^{\frac12} \|\partial_y u_1\|_2^{\frac12} \|\omega_x\|_2^{\frac12} \|\omega_{xx}\|_2^{\frac12} \\
&\le C\|\omega_{xx}\|_2^{\frac32} \|u_1\|_2^{\frac12} \|\partial_y u_1\|_2^{\frac12} \|\omega_x\|_2^{\frac12}
\end{aligned}
$$

$$
\begin{aligned}
&\le \frac{1}{48}\|\omega_{xx}\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2.\\
\left|\int u_1\omega_y\omega_{xy}\right| &\le C\|\omega_{xy}\|_2\|u_1\|_2^{\frac12}\|\|\partial_y u_1\|_2^{\frac12}\|\partial_{xy}\omega\|_2^{\frac12}\|\partial_y\omega\|_2^{\frac12}\\
&\le C\|\omega_{xy}\|_2^{\frac32}\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\nabla\omega\|_2^{\frac12}\\
&\le \frac{1}{48}\|\nabla\omega_x\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
\left|\int(\partial_x u_2+\partial_y u_1)\omega_x\omega_y\right| &\le C(\|\partial_x u_2\|_2+\|\partial_y u_1\|_2)\|\omega_x\|_2^{\frac12}\|\omega_{xy}\|_2^{\frac12}\|\omega_y\|_2^{\frac12}\|\omega_{xy}\|_2^{\frac12}\\
&\le C(\|\partial_x u_2\|_2+\|\partial_y u_1\|_2)\|\nabla\omega\|_2\|\nabla\omega_x\|_2\\
&\le \frac{1}{48}\|\nabla\omega_x\|_2^2+C(\|\partial_x u_2\|_2^2+\|\partial_y u_1\|_2^2)\|\nabla\omega\|_2^2.
\end{aligned}
$$

To estimate J_6, we first integrate by part to obtain

$$
J_6=2\int\nabla\omega\cdot\nabla\Omega=-2\int\omega_{xx}\Omega+2\int\omega_y\Omega_y.
$$

The terms on the right can be bounded as follows.

$$
\begin{aligned}
&\left|\int\omega_{xx}\Omega\right|\le\|\omega_{xx}\|_2\|\Omega\|_2\le\frac12\|\nabla\omega_x\|_2^2+C\|\Omega\|_2^2,\\
&\int\omega_y\Omega_y=\int(\omega_y\partial_{xy}u_2-\omega_y\partial_{yy}u_1),\\
&\left|\int\omega_y\partial_{xy}u_2\right|\le C\|\nabla\omega\|_2\|\nabla\partial_x u_2\|_2,\\
&\left|\int\omega_y\partial_{yy}u_1\right|\le C\|\nabla\omega\|_2\|\nabla\partial_y u_1\|_2.
\end{aligned}
$$

Combining the estimates above, together with Gronwall's inequalities, we obtain

$$
\begin{aligned}
&\|\Omega\|_2^2+\|j\|_2^2+\|\nabla\omega\|_2^2\\
&\quad+\int_0^t\left(\|\nabla\partial_y u_1\|_2^2+\|\nabla\partial_x u_2\|_2^2+\|\partial_x j\|_{L^2}^2+\|\nabla\omega_x\|_2^2+2\|\nabla\omega\|_2^2\right)d\tau\le C
\end{aligned}
$$

for any $t\le T$, where C depends on T and the initial H^1-norm. Especially, (2.2) is proven. This completes the proof of Proposition 2.1. □

2.2 Global H^2 bound and the proof of Theorem 1.1

This subsection proves Theorem 1.1 by establishing the global H^2 bound for the solution.

Proof of Theorem 1.1. As we explained before, it suffices to establish the global H^2-bound in order to prove Theorem 1.1. The rest of this proof establishes the global H^2-bound.

Taking the L^2 inner product of (2.3) with $\nabla\Omega$ and (2.5) with ∇j, and integrating by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2)+\|\Delta\partial_y u_1\|_2^2+\|\Delta\partial_x u_2\|_2^2+\|\nabla\partial_x j\|_2^2\\&\quad=L_1+L_2+L_3+L_4+L_5+L_6,\end{aligned}\tag{2.6}$$

where

$$L_1=-\int\nabla\Omega\cdot\nabla u\cdot\nabla\Omega\,dxdy,\qquad L_2=-\int\nabla j\cdot\nabla u\cdot\nabla j\,dxdy,$$
$$L_3=2\int\nabla\Omega\cdot\nabla b\cdot\nabla j\,dxdy,\qquad L_4=2\int\nabla[\partial_x b_1(\partial_x u_2+\partial_y u_1)]\cdot\nabla j\,dxdy,$$
$$L_5=-2\int\nabla[\partial_x u_1(\partial_x b_2+\partial_y b_1)]\cdot\nabla j\,dxdy,\qquad L_6=\int\Delta\Omega\Delta\omega\,dxdy.$$

Applying ∇ to (2.4) and taking the L^2-inner product with $\Delta\omega$, and integrating by parts, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta\omega\|_2^2+2\|\Delta\omega_x\|_2^2+\|\Delta\omega\|_2^2&=\int\Delta\Omega\Delta\omega-\int\Delta(u\cdot\nabla\omega)\Delta\omega\\&\equiv L_6+L_7.\end{aligned}\tag{2.7}$$

Adding (2.6) and (2.7) yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2+\|\Delta\omega\|_2^2)+\|\Delta\partial_y u_1\|_2^2\\&\qquad+\|\Delta\partial_x u_2\|_2^2+\|\nabla\partial_x j\|_2^2+2\|\Delta\omega_x\|_2^2+\|\Delta\omega\|_2^2\\&=L_1+L_2+L_3+L_4+L_5+2L_6+L_7.\end{aligned}$$

We now estimate L_1 through L_7. We further split L_1 into 4 terms.

$$\begin{aligned}L_1&=-\int\nabla\Omega\cdot\nabla u\cdot\nabla\Omega\,dxdy\\&=-\int\big(\partial_x u_1(\partial_x\Omega)^2+\partial_x u_2\partial_x\Omega\partial_y\Omega+\partial_y u_1\partial_x\Omega\partial_y\Omega+\partial_y u_2(\partial_y\Omega)^2\big)\\&=L_{11}+L_{12}+L_{13}+L_{14}.\end{aligned}$$

Due to $\Omega=\nabla\times u$, we have

$$\partial_{xx}\Omega=\Delta\partial_x u_2,\quad \partial_{yy}\Omega=-\Delta\partial_y u_1,\quad \partial_{xy}\Omega=\Delta\partial_x u_2.$$

Therefore,

$$L_{11}=-\int\partial_x u_1(\partial_{xx}u_2)^2-\int\partial_x u_1(\partial_{xy}u_1)^2+2\int\partial_x u_1\partial_{xx}u_2\partial_{xy}u_1.$$

Integration by parts yields

$$\begin{aligned}\int \partial_x u_1 (\partial_{xx} u_2)^2 &= -\int \partial_{xx} u_1 \partial_{xx} u_2 \partial_x u_2 - \int \partial_x u_1 \partial_{xxx} u_2 \partial_x u_2\\ &\equiv L_{111} + L_{112},\end{aligned}$$

which can be bounded as

$$\begin{aligned}L_{111} &\le C\|\partial_{xx}u_2\| \|\partial_{xx}u_1\|_2^{\frac{1}{2}} \|\partial_{xxy}u_1\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_{xx}u_2\|_2^{\frac{1}{2}}\\ &\le C\|\partial_{xx}u_2\|_2^{\frac{3}{2}} \|\partial_{xx}u_1\|_2^{\frac{1}{2}} \|\Delta\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}}\\ &\le \frac{1}{48}\|\Delta\partial_x u_2\|_2^2 + C\|\partial_{xx}u_2\|_2^2 \|\partial_{xx}u_1\|_2^{\frac{2}{3}} \|\partial_x u_2\|_2^{\frac{2}{3}}.\end{aligned}$$

$$\begin{aligned}L_{112} &\le C\|\partial_{xxx}u_2\|_2 \|\partial_x u_1\|_2^{\frac{1}{2}} \|\partial_{xy}u_1\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_{xx}u_2\|_2^{\frac{1}{2}}\\ &\le C\|\Delta\partial_x u_2\|_2 \|\Omega\|_2^{\frac{1}{2}} \|\nabla\Omega\|_2 \|\partial_x u_2\|_2^{\frac{1}{2}}\\ &\le \frac{1}{48}\|\Delta\partial_x u_2\|_2^2 + C\|\Omega\|_2 \|\partial_x u_2\|_2 \|\nabla\Omega\|_2^2.\end{aligned}$$

By Lemma 2.1,

$$\begin{aligned}L_{12} &= \int \partial_x u_2 \partial_x\Omega \partial_y\Omega\\ &\le C\|\partial_x u_2\|_2 \|\partial_x\Omega\|_2^{\frac{1}{2}} \|\partial_{xx}\Omega\|_2^{\frac{1}{2}} \|\partial_y\Omega\|_2^{\frac{1}{2}} \|\partial_{yy}\Omega\|_2^{\frac{1}{2}}\\ &\le C\|\partial_x u_2\|_2 \|\nabla\Omega\|_2 \|\partial_{xx}\Omega\|_2^{\frac{1}{2}} \|\partial_{yy}\Omega\|_2^{\frac{1}{2}}\\ &\le \|\partial_{xx}\Omega\| \|\partial_{yy}\Omega\| + C\|\partial_x u_2\|_2^2 \|\nabla\Omega\|_2^2\\ &\le \frac{1}{48}(\|\partial_{xx}\Omega\|_2^2 + \|\partial_{yy}\Omega\|_2^2) + C\|\partial_x u_2\|_2^2 \|\nabla\Omega\|_2^2.\end{aligned}$$

$$\begin{aligned}L_{13} &\le \left|\int \partial_y u_1 \partial_x\Omega \partial_y\Omega\right|\\ &\le C\|\partial_y u_1\|_2^{\frac{1}{2}} \|\partial_{xy}u_1\|_2^{\frac{1}{2}} \|\partial_x\Omega\|_2 \|\partial_y\Omega\|_2^{\frac{1}{2}} \|\partial_{yy}\Omega\|_2^{\frac{1}{2}}\\ &\le C\|\partial_{yy}\Omega\|_2 \|\partial_{xy}u_1\|_2 \|\partial_y\Omega\|_2 + C\|\partial_y u_1\|_2 \|\nabla\Omega\|_2^2\\ &\le \frac{1}{48}\|\partial_{yy}\Omega\|_2^2 + C(\|\partial_y u_1\|_2^2 + \|\partial_{xy}u_1\|_2^2) \|\nabla\Omega\|_2^2.\end{aligned}$$

$$\begin{aligned}L_{14} &\le \left|2\int u_2 \partial_y\Omega \partial_{yy}\Omega\right|\\ &\le C\|\partial_{yy}\Omega\|_2 \|u_2\|_2^{\frac{1}{2}} \|\partial_x u_2\|_2^{\frac{1}{2}} \|\partial_y\Omega\|_2^{\frac{1}{2}} \|\partial_{yy}\Omega\|_2^{\frac{1}{2}}\\ &\le C\|\partial_{yy}\Omega\|_2^{\frac{3}{2}} \|u_2\|_2^{\frac{1}{2}} \|\Omega\|_2^{\frac{1}{2}} \|\nabla\Omega\|_2^{\frac{1}{2}}\end{aligned}$$

$$\leq \frac{1}{48}\|\partial_{yy}\Omega\|_2^2+C\|u_2\|_2^2\|\Omega\|_2^2\|\nabla\Omega\|_2^2.$$

To estimate L_2, we write it out explicitly as

$$\begin{aligned}
L_2 &= -\int \nabla j\cdot\nabla u\cdot\nabla j\,dxdy\\
&= -\int \big(\partial_x u_1(\partial_x j)^2+\partial_y u_1\partial_x j\partial_y j+\partial_y u_2(\partial_y j)^2+\partial_x u_2\partial_x j\partial_y j\big)\\
&= L_{21}+L_{22}+L_{23}+L_{24}.
\end{aligned}$$

The terms on the right can be bounded as follows.

$$\begin{aligned}
L_{21} &\leq C\|\partial_x j\|_2\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xx}j\|_2^{\frac12}\\
&\leq C\|\Omega\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\nabla j\|^{\frac32}\|\nabla\partial_x j\|_2^{\frac12}\\
&\leq \frac{1}{48}\|\nabla\partial_x j\|_2^2+C\|\partial_{xy}u_1\|_2^{\frac23}\|\Omega\|_2^{\frac23}\|\nabla j\|_2^2.
\end{aligned}$$

$$\begin{aligned}
L_{22} &\leq C\|\partial_y u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}\|\partial_y j\|_2\\
&\leq C\|\Omega\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\nabla j\|_2^{\frac32}\|\nabla\partial_x j\|_2^{\frac12}.
\end{aligned}$$

$$\begin{aligned}
L_{23} &\leq C\|\partial_y j\|_2\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_y j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}\\
&\leq C\|\nabla j\|_2^{\frac32}\|\nabla\partial_x j\|_2^{\frac12}\|\Omega\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\\
&\leq \frac{1}{48}\|\nabla\partial_x j\|_2^2+\|\Omega\|_2^{\frac23}\|\partial_{xy}u_1\|_2^{\frac23}\|\nabla j\|_2^2
\end{aligned}$$

$$\begin{aligned}
L_{24} &\leq C\|\partial_x u_2\|_2\|\partial_y j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}\\
&\leq C\|\Omega\|_2\|\nabla j\|_2\|\nabla\partial_x j\|_2.
\end{aligned}$$

We now turn to L_3. Again we write it out as

$$\begin{aligned}
L_3 &= \int \partial_x\Omega\partial_x b_1\partial_x j+\partial_x\Omega\partial_x b_2\partial_y j+\partial_y\Omega\partial_y b_1 j_x+\partial_y\Omega\partial_y b_2\partial_y j\\
&\equiv L_{31}+L_{32}+L_{33}+L_{34}.
\end{aligned}$$

The terms on the right can be bounded as follows.

$$\begin{aligned}
L_{31} &\leq \left|\int \partial_x\Omega\partial_x b_1\partial_x j\right|\\
&\leq C\|\partial_x\Omega\|_2\|\partial_x b_1\|_2^{\frac12}\|\partial_{xx}b_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}\\
&\leq C\|\partial_x\Omega\|_2\|\partial_x b_1\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_{xy}j\|_2^{\frac12}
\end{aligned}$$

$$\leq C\|\partial_x j\|_2\|\nabla\partial_x j\|_2+\|\partial_x j\|\|\partial_x b_1\|\|\partial_x\Omega\|_2^2$$
$$\leq \frac{1}{48}\|\nabla\partial_x j\|_2^2+C(\|\partial_x b_1\|_2^2+\|\partial_x j\|_2^2+1)(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2).$$

$$L_{32}\leq \frac{1}{48}\|\nabla j_x\|_2^2+C(\|\partial_x b_2\|_2^2+\|\partial_x j\|^2+1)(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2).$$

$$\begin{aligned}L_{33}&\leq C\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2\\&\leq C\|j\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2\\&\leq \frac{1}{48}\|\nabla\partial_x j\|_2^2+C\|\partial_x j\|_2^2\|\nabla j\|_2^2+C\|j\|_2\|\nabla\Omega\|_2^2.\end{aligned}$$

$$\begin{aligned}L_{34}&\leq C\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2\\&\leq C\|j\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2\\&\leq \frac{1}{48}\|\nabla\partial_x j\|_2^2+C\|\partial_x j\|_2^2\|\nabla j\|_2^2+C\|j\|_2\|\nabla\Omega\|_2^2.\end{aligned}$$

We now estimate L_4.

$$\begin{aligned}L_4&=2\int\nabla[\partial_x b_1(\partial_x u_2+\partial_y u_1)]\cdot\nabla j\,dxdy\\&=2\int\partial_x[\partial_x b_1(\partial_x u_2+\partial_y u_1)]j_x+\partial_y[\partial_x b_1(\partial_x u_2+\partial_y u_1)]j_y\,dxdy\\&\equiv L_{41}+L_{42}.\end{aligned}$$

We bound L_{41} and L_{42} as follows.

$$\begin{aligned}L_{41}&\leq \left|-2\int\partial_x b_1(\partial_x u_2+\partial_y u_1)\partial_{xx} j\right|\\&\leq C(\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xx} b_1\|_2^{\frac{1}{2}}\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xy} u_2\|_2^{\frac{1}{2}}\\&\qquad+C\|\partial_x b_1\|_2^{\frac{1}{2}}\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy} u_1\|_2^{\frac{1}{2}})\|\partial_{xx} j\|_2\\&\leq C\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\Omega\|_2^{\frac{1}{2}}\|\Omega_y\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2\\&\leq \frac{1}{48}\|\nabla\partial_x j\|_2^2+C\|\Omega\|_2\|j\|_2(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2).\end{aligned}$$

L_{42} can be more explicitly written as

$$\begin{aligned}L_{42}&=2\int(\partial_{xy} b_1\partial_x u_2+\partial_x b_1\partial_{xy} u_2+\partial_{xy} b_1\partial_y u_1+\partial_x b_1\partial_{yy} u_1)\partial_y j\,dxdy\\&\equiv L_{421}+L_{422}+L_{423}+L_{424}.\end{aligned}$$

The bounds for the terms on the right are given as follows.

$$L_{421}\leq C\|\partial_x u_2\|_2\|\partial_{xy} b_1\|_2^{\frac{1}{2}}\|\partial_{xyy} b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2^{\frac{1}{2}}\|\partial_{xy} j\|_2^{\frac{1}{2}}$$

$$
\begin{aligned}
&\le C\|\Omega\|_2\|\nabla j\|_2\|\nabla\partial_x j\|_2\\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\Omega\|_2^2\|\nabla j\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
L_{422} &\le C\|\partial_x b_1\|_2^{\frac12}\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_{xy} u_2\|_2\|\partial_y j\|_2^{\frac12}\|\partial_{xy} j\|_2^{\frac12}\\
&\le C\|j\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\partial_x\Omega\|\|\nabla j\|_2^{\frac12}\|\nabla\partial_x j\|_2^{\frac12}\\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\partial_x j\|_2^2\|\nabla j\|_2^2 + C\|j\|_2\|\nabla\Omega\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
L_{423} &\le C\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_{xyy} b_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\partial_y j\|_2\\
&\le C\|\partial_x j\|_2^{\frac12}\|\nabla\partial_x j\|_2^{\frac12}\|\Omega\|_2^{\frac12}\|\partial_x\Omega\|_2^{\frac12}\|\nabla j\|_2\\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\Omega\|_2\|\nabla j\|_2^2 + C\|\partial_x j\|_2^2\|\nabla\Omega\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
L_{424} &\le C\|\partial_y j\|_2\|\partial_x b_1\|_2^{\frac12}\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_{yy} u_1\|_2^{\frac12}\|\partial_{xyy} u_1\|_2^{\frac12}\\
&\le C\|\nabla j\|_2\|j\|_2^{\frac12}\|\partial_x j\|_2^{\frac12}\|\nabla\Omega\|_2^{\frac12}\|\partial_{yy}\Omega\|_2^{\frac12}\\
&\le \frac{1}{48}\|\partial_{yy}\Omega\|_2^2 + C\|\partial_x j\|_2^2\|\nabla\Omega\|_2^2 + C\|j\|_2\|\nabla j\|_2^2.
\end{aligned}
$$

We now estimate L_5. More explicitly, L_5 is written as

$$
\begin{aligned}
L_5 &= -2\int \nabla[\partial_x u_1(\partial_x b_2 + \partial_y b_1)]\cdot\nabla j\,dxdy\\
&= -2\int \partial_x[\partial_x u_1(\partial_x b_2+\partial_y b_1)]\partial_x j + \partial_y[(\partial_x u_1(\partial_x b_2+\partial_y b_1)]\partial_y j\,dxdy\\
&\equiv L_{51} + L_{52}.
\end{aligned}
$$

L_{51} is bounded as follows.

$$
\begin{aligned}
L_{51} \le\ & C\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\partial_x b_2\|_2^{\frac12}\|\partial_{xx} b_2\|_2^{\frac12}\|\partial_{xx} j\|_2\\
&\quad + C\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy} u_1\|_2^{\frac12}\|\partial_y b_1\|_2^{\frac12}\|\partial_{xy} b_1\|_2^{\frac12}\|\partial_{xx} j\|_2\\
\le\ & C\|\Omega\|_2^{\frac12}\|\nabla\Omega\|_2^{\frac12}\|j\|_2^{\frac12}\|\nabla j\|_2^{\frac12}\|\nabla\partial_x j\|_2\\
\le\ & \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\Omega\|_2\|j\|_2(\|\nabla\Omega\|_2^2 + \|\nabla j\|_2^2).
\end{aligned}
$$

L_{52} contains four terms.

$$
\begin{aligned}
L_{52} &= -2\int (\partial_{xy} u_1\partial_x b_2 + \partial_x u_1\partial_{xy} b_2 + \partial_{xy} u_1\partial_y b_1 + \partial_x u_1\partial_{yy} b_1)\partial_y j\,dxdy\\
&\equiv L_{521} + L_{522} + L_{523} + L_{524}.
\end{aligned}
$$

These terms are estimated as follows.

$$L_{521} \le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C(\|\partial_x j\|_2^2 + \|j\|_2^2)(\|\nabla j\|_2^2 + \|\nabla\Omega\|_2^2).$$

$$L_{522} \le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\Omega\|_2^{\frac{2}{3}}\|\partial_{xy}u_1\|_2^{\frac{2}{3}}\|\nabla j\|_2^2.$$

$$\begin{aligned}
L_{523} &\le C\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{xyy}u_1\|_2^{\frac{1}{2}}\|\partial_y b_1\|_2^{\frac{1}{2}}\|\partial_{xy}b_1\|_2^{\frac{1}{2}}\|\partial_y j\|_2 \\
&\le C\|\Omega_{yy}\|_2^{\frac{1}{2}}\|\partial_x j\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\|j\|_2^{\frac{1}{2}}\|\nabla j\|_2 \\
&\le \frac{1}{48}\|\partial_{yy}\Omega\|_2^2 + C\|\partial_x j\|_2^2\|\nabla\Omega\|_2^2 + C\|j\|_2\|\nabla j\|_2^2.
\end{aligned}$$

$$\begin{aligned}
L_{524} &\le C\|\partial_y j\|_2\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\partial_{yy}b_1\|_2^{\frac{1}{2}}\|\partial_{xyy}b_1\|_2^{\frac{1}{2}} \\
&\le C\|\nabla j\|_2\|\Omega\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\|\nabla j\|_2^{\frac{1}{2}}\|\nabla\partial_x j\|_2^{\frac{1}{2}} \\
&\le \frac{1}{48}\|\nabla\partial_x j\|_2^2 + C\|\partial_{xy}u_1\|_2^2\|\nabla j\|_2^2 + \|\Omega\|_2\|\nabla j\|_2^2.
\end{aligned}$$

L_6 can be easily bounded.

$$L_6 = \int \Delta\Omega\Delta\omega = \int \Omega_{xx}\Delta\omega + \int \Omega_{yy}\Delta\omega$$

with

$$\int \Omega_{xx}\Delta\omega = -\int \Omega_x\Delta\omega_x \le \|\nabla\Omega\|_2\|\Delta\omega_x\|_2, \qquad \left|\int \Omega_{yy}\Delta\omega\right| \le \|\Omega_{yy}\|_2\|\Delta\omega\|_2.$$

We now estimate the last term L_7.

$$L_7 = -\int \Delta(u\cdot\nabla\omega)\Delta\omega = -\int \Delta(u_1\partial_1\omega + u_2\partial_2\omega)\Delta\omega \equiv L_{71} + L_{72}.$$

We first split L_{71} and L_{72} each into two terms.

$$\begin{aligned}
L_{71} &= -\int \partial_{xx}(u_1\partial_x\omega + u_2\partial_y\omega)\Delta\omega = L_{711} + L_{712}. \\
L_{72} &= -\int \partial_{yy}(u_1\partial_x\omega + u_2\partial_y\omega)\partial_{xx}\omega - \int \partial_{yy}(u_1\partial_x\omega + u_2\partial_y\omega)\partial_{yy}\omega = L_{721} + L_{722}.
\end{aligned}$$

These terms are bounded as follows.

$$\begin{aligned}
|L_{711}| &= \left|-\int \partial_x(u_1\partial_x\omega)\Delta\omega_x\right| \\
&\le \left|-\int \partial_x u_1\partial_x\omega\Delta\omega_x\right| + \left|\int u_1\partial_{xx}\omega\Delta\omega_x\right|
\end{aligned}$$

$$\begin{aligned}
&\le C\|\Delta\omega_x\|_2\|\partial_x u_1\|_2^{\frac12}\|\partial_{xy}u_1\|_2^{\frac12}\|\partial_x\omega\|_2^{\frac12}\|\partial_{xx}\omega\|_2^{\frac12}\\
&\quad +C\|\Delta\omega_x\|_2\|\partial_{xx}\omega\|_2^{\frac12}\|\partial_{xxx}\omega\|_2^{\frac12}\|u_1\|_2^{\frac12}\|\partial_y u_1\|_2^{\frac12}\\
&\le C\|\Delta\omega_x\|_2\|\Delta\omega\|_2^{\frac12}\|\Omega\|_2^{\frac12}\|\nabla\Omega\|_2^{\frac12}\|\nabla\omega\|_2^{\frac12}+C\|\Delta\omega_x\|_2^{\frac32}\|\nabla\omega_x\|_2^{\frac12}\|u_1\|_2^{\frac12}\|\Omega\|_2^{\frac12}\\
&\le \frac{1}{48}\|\Delta\omega_x\|_2^2+C\|\Omega\|_2^2\|\nabla\Omega\|_2^2+\|\nabla\omega\|_2^2\|\Delta\omega\|_2^2+C\|u_1\|_2^2\|\nabla\omega_x\|_2^2\|\Omega\|_2^2.
\end{aligned}$$

$$\begin{aligned}
|L_{712}|&=\left|-\int\partial_x u_2\partial_y\omega\Delta\omega_x-\int u_2\partial_{xy}\omega\Delta\omega_x\right|\\
&\le C\|\Delta\omega_x\|_2\|\partial_x u_2\|_2^{\frac12}\|\partial_{xy}u_2\|_2^{\frac12}\|\partial_y\omega\|_2^{\frac12}\|\partial_{xy}\omega\|_2^{\frac12}\\
&\quad +C\|\Delta\omega_x\|_2\|u_2\|_2^{\frac12}\|\partial_x u_2\|_2^{\frac12}\|\partial_{xy}\omega\|_2^{\frac12}\|\partial_{xyy}\omega\|_2^{\frac12}.
\end{aligned}$$

$$L_{721}=\int\partial_{yy}(u_1\partial_x\omega+u_2\partial_y\omega)\partial_{xx}\omega=\int\partial_{xx}(u_1\partial_x\omega+u_2\partial_y\omega)\partial_{yy}\omega.$$

Obviously L_{721} admits the same bound as that for L_{711},

$$|L_{721}|\le\frac{1}{48}\|\Delta\omega_x\|_2^2+C\|\Omega\|_2^2\|\nabla\Omega\|_2^2+\|\nabla\omega\|_2^2\|\Delta\omega\|_2^2+C\|u_1\|_2^2\|\nabla\omega_x\|_2^2\|\Omega\|_2^2.$$

To estimate L_{722}, we write it out explicitly and integrate by parts,

$$\begin{aligned}
L_{722}&=\int\partial_{yy}(u_1\partial_x\omega+u_2\partial_y\omega)\partial_{yy}\omega\\
&=\int\partial_y(\partial_y u_1\partial_x\omega+u_1\partial_{xy}\omega)\partial_{yy}\omega+\int\partial_y(\partial_y u_2\partial_2\omega+u_2\partial_{yy}\omega)\partial_{yy}\omega\\
&=\int[\partial_{yy}u_1\partial_x\omega+2\partial_y u_1\partial_{xy}\omega+u_1\partial_{xyy}\omega]\partial_{yy}\omega\\
&\qquad+\int[\partial_{yy}u_2\partial_y\omega+2\partial_y u_2\partial_{yy}\omega+u_2\partial_{yyy}\omega]\partial_{yy}\omega.
\end{aligned}$$

The terms on the right can be bounded as follows.

$$\begin{aligned}
&\left|\int\partial_{yy}u_1\partial_x\omega\partial_{yy}\omega\right|\\
&\le C\|\partial_x\omega\|_2\|\partial_{yy}u_1\|_2^{\frac12}\|\partial_{yyy}u_1\|_1^{\frac12}\|\partial_{yy}\omega\|_2^{\frac12}\|\partial_{xyy}\omega\|_2^{\frac12}\\
&\le C\|\partial_x\omega\|_2\|\nabla\partial_y u_1\|_2^{\frac12}\|\Delta\partial_y u_1\|_2^{\frac12}\|\Delta\omega\|_2^{\frac12}\|\Delta\partial_x\omega\|_2^{\frac12}\\
&\le\frac{1}{48}\|\Delta\partial_y u_1\|_2^2+\frac{1}{48}\|\Delta\omega_x\|_2^2+C\|\omega_x\|_2^2[\|\nabla\partial_y u_1\|_2^2+\|\Delta\omega\|_2^2)\\
&\le\frac{1}{48}\|\Delta\partial_y u_1\|_2^2+\frac{1}{48}\|\Delta\omega_x\|_2^2+C\|\omega_x\|_2^2(\|\nabla\Omega\|_2^2+\|\Delta\omega\|_2^2).
\end{aligned}$$

$$\left|\int\partial_y u_1\partial_{xy}\omega\partial_{yy}\omega\right|$$

$$
\begin{aligned}
&\le C\|\partial_{yy}\omega\|_2\|\partial_{xy}\omega\|_2^{\frac{1}{2}}\|\partial_{xyy}\omega\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{xy}u_1\|_2^{\frac{1}{2}}\\
&\le C\|\Delta\omega\|_2\|\nabla\omega_x\|_2^{\frac{1}{2}}\|\Delta\omega_x\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\nabla\Omega\|_2^{\frac{1}{2}}\\
&\le \frac{1}{48}\|\Delta\omega_x\|_2^2+C\|\nabla\omega_x\|_2^2\|\nabla\Omega\|_2^2+C\|\partial_y u_1\|_2\|\Delta\omega\|_2^2
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int u_1\partial_{xyy}\omega\partial_{yy}\omega\right|\\
&\le C\|\partial_{xyy}\omega\|\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\partial_{yy}\omega\|_2^{\frac{1}{2}}\|\partial_{xyy}\omega\|_2^{\frac{1}{2}}\\
&\le C\|\Delta\omega_x\|_2^{\frac{3}{2}}\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\|\Delta\omega\|_2^{\frac{1}{2}}\\
&\le \frac{1}{48}\|\Delta\omega_x\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\Delta\omega\|_2^2.
\end{aligned}
$$

$$
\begin{aligned}
&\left|\int \partial_{yy}u_2\partial_y\omega\partial_{yy}\omega\right|\\
&\le C\|\partial_{yy}u_2\|_2\|\partial_y\omega\|_2^{\frac{1}{2}}\|\omega_{yy}\|_2\|\omega_{xyy}\|_2^{\frac{1}{2}}\\
&\le \|\omega_y\|_2\|\Delta\omega_x\|_2+\|\nabla\partial_y u_1\|_2^2\|\Delta\omega\|_2^2\\
&\le \frac{1}{48}\|\Delta\omega_x\|_2^2+C\|\nabla\omega\|_2^2+C\|\nabla\partial_y u_1\|_2^2\|\Delta\omega\|_2^2.
\end{aligned}
$$

Integration by parts yields

$$\int\partial_y u_2\partial_{yy}\omega\partial_{yy}\omega=-\int\partial_x u_1\partial_{yy}\omega\partial_{yy}\omega=2\int u_1\partial_{yy}\omega\partial_{xyy}\omega$$

and

$$\int u_2\partial_{yyy}\omega\partial_{yy}\omega=\frac{1}{2}\int u_2\partial_y[\partial_{yy}\omega]^2=\int u_1\partial_{xyy}\omega\partial_{yy}\omega,$$

which can be bounded as

$$
\begin{aligned}
&\left|\int u_1\partial_{xyy}\omega\partial_{yy}\omega\right|\\
&\le C\|\partial_{xyy}\omega\|_2\|\partial_{yy}\omega\|_2^{\frac{1}{2}}\|\partial_{xyy}\omega\|_2^{\frac{1}{2}}\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\\
&\le \|\Delta\omega_x\|_2^{\frac{3}{2}}\|\Delta\omega\|_2^{\frac{1}{2}}\|u_1\|_2^{\frac{1}{2}}\|\partial_y u_1\|_2^{\frac{1}{2}}\\
&\le \|\Delta\omega_x\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\Delta\omega\|_2^2.
\end{aligned}
$$

Collecting the estimates above and applying Gronwall's inequality, we obtain the desired global H^2-bound. This completes the proof for the global H^2-bound and thus the proof of Theorem 1.1. □

3 Proof of Theorem 1.2

This section proves Theorem 1.2, which assesses the global existence and regularity of solutions to (1.5). Again the main task is to prove the global H^2-bound for the solution.

First of all, we can easily prove the following global L^2-bound.

Lemma 3.1. *Assume that (u_0,b_0,ω_0) satisfies the condition stated in Theorem* 1.2*. Let (u,b,ω) be the corresponding solution of* (1.5)*. Then, (u,b,ω) obeys the following global L^2-bound,*

$$\begin{aligned}&\|u(t)\|_{L^2}^2+\|b(t)\|_{L^2}^2+\|\omega(t)\|_{L^2}^2+2\int_0^t\|\partial_x u_1,\partial_y u_1\|_{L^2}^2d\tau+2\int_0^t\|\partial_x b(\tau)\|_{L^2}^2d\tau\\&\quad+2\int_0^t\|\partial_x\omega(\tau)\|_{L^2}^2d\tau\le C(\|u_0\|_{L^2}^2,\|b_0\|_{L^2}^2,\|\omega_0\|_2^2)\end{aligned}$$

for any $t\ge 0$.

The rest of this section is divided into two subsections. The first subsection proves the global H^1-bound while the second proves the global H^2-bound, which leads to the proof of Theorem 1.2.

3.1 Global H^1 bound

This subsection proves that the solution of (1.5) is globally bounded in H^1-norm. More precisely, we prove the following proposition.

Proposition 3.1. Assume that (u_0,b_0,ω_0) satisfies the condition stated in Theorem 1.2. Let (u,b,ω) be the corresponding solution of (1.5). Then (u,b,ω) satisfies, for any $T>0$,

$$u,b,\omega\in C([0,T];H^1).$$

Proof. To prove the global H^1-bound, we start with the equations for $\Omega=\nabla\times u$, $\nabla\omega$ and $j=\nabla\times b$,

$$\begin{aligned}&\Omega_t+u\cdot\nabla\Omega=-\partial_{xxy}u_1-\partial_{yyy}u_1+(b\cdot\nabla)j-\Delta\omega, &(3.1)\\&\partial_t\nabla\omega+\nabla(u\cdot\nabla\omega)+2\nabla\omega=\nabla\omega_{xx}+\nabla\Omega, &(3.2)\\&j_t+u\cdot\nabla j=\partial_{xx}j+b\cdot\nabla\Omega+2\partial_x b_1(\partial_x u_2+\partial_y u_1)-2\partial_x u_1(\partial_x b_2+\partial_y b_1), &(3.3)\\&\nabla\cdot u=0,\quad \nabla\cdot b=0.\end{aligned}$$

Taking the inner product of these equations with $(\Omega,\nabla\omega,j)$, integrating by parts and applying $\nabla\cdot u=0$ and $\nabla\cdot b=0$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2+\|j\|_{L^2}^2+\|\nabla\omega\|_2^2\right)+\|(\nabla\partial_x u_1,\nabla\partial_y u_1)\|_2^2\\&\quad+\|\partial_x j\|_{L^2}^2+\|\nabla\omega_x\|_2^2+2\|\nabla\omega\|_2^2\end{aligned}$$

$$
\begin{aligned}
&= 2\int \left[\partial_x b_1(\partial_x u_2+\partial_y u_1)j-\partial_x u_1(\partial_x b_2+\partial_y b_1)\right] j dxdy \\
&\qquad -\int \nabla\omega\cdot\nabla u\cdot\nabla\omega+2\int \nabla\omega\cdot\nabla\Omega. \\
&= J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}
$$

The terms J_1,J_2,J_3,J_4 can be estimated in a similar fashion as in the previous section and their corresponding bounds are

$$
\begin{aligned}
J_1 &\le \frac{1}{48}\|\partial_x j\|_2^2+C\|\partial_x b\|_2^2\|\Omega\|_2^2+C\|u_2\|\|j\|_2\|\Omega\|_2, \\
J_2 &\le \frac{1}{48}\left[\|\partial_{yy}u_1\|_2^2+\|\partial_x j\|_2^2\right]+C(\|\partial_x b_1\|_2^2+\|\partial_y u_1\|_2^2)\|j\|_2^2, \\
J_3 &\le \frac{1}{48}\|\partial_x j\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2, \\
J_4 &\le \frac{1}{48}\|\partial_x j\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|j\|_2^2.
\end{aligned}
$$

We focus on the bounds for J_5 and J_6. Writing out the terms in J_5 explicitly and applying $\nabla\cdot u=0$, we obtain

$$
J_5=-\int \nabla\omega\cdot\nabla u\cdot\nabla\omega=2\int u_1\omega_{xx}\omega_x-2\int u_1\omega_y\omega_{xy}-\int(\partial_x u_2+\partial_y u_1)\omega_x\omega_y.
$$

The terms on the right are bounded as follows.

$$
\begin{aligned}
\left|\int u_1\omega_{xx}\omega_x\right| &\le \frac{1}{48}\|\omega_{xx}\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2. \\
\left|\int u_1\omega_y\omega_{xy}\right| &\le \frac{1}{48}\|\nabla\omega_x\|_2^2+C\|u_1\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2. \\
\left|\int \partial_y u_1\omega_x\omega_y\right| &\le \frac{1}{48}\|\nabla\omega_x\|_2^2+C\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2.
\end{aligned}
$$

By integration by parts,

$$
\int \partial_x u_2\omega_x\omega_y=-\int \partial_{xy}u_2\omega_x\omega-\int \partial_x u_2\omega_{xy}\omega.
$$

The two terms on the right admit the following bounds.

$$
\begin{aligned}
\left|-\int \partial_{xy}u_2\omega_x\omega\right| &\le C\|\partial_{xy}u_2\|_2\|\omega_x\|_2^{\frac12}\|\omega_{xy}\|_2^{\frac12}\|\omega\|_2^{\frac12}\|\omega_x\|_2^{\frac12} \\
&\le C\|\nabla\partial_x u_1\|_2\|\omega_x\|_2\|\nabla\omega_x\|_2^{\frac12}\|\omega\|_2^{\frac12} \\
&\le C\|\nabla\partial_x u_1\|_2^2+\|\nabla\omega_x\|_2\|\omega\|_2\|\partial_x\omega\|_2\|\nabla\omega\| \\
&\le C\|\nabla\partial_x u_1\|_2^2+\|\nabla\omega_x\|_2^2+\|\omega\|_2^2\|\omega_x\|_2^2\|\nabla\omega\|_2^2.
\end{aligned}
$$

$$\begin{aligned}\left|-\int \partial_x u_2 \omega_{xy}\omega\right| &\le C\|\omega_{xy}\|_2\|\partial_x u_2\|_2^{\frac{1}{2}}\|\partial_{xy}u_2\|_2^{\frac{1}{2}}\|\omega\|_2^{\frac{1}{2}}\|\omega_x\|_2^{\frac{1}{2}}\\ &\le C\|\nabla\omega_x\|\|\Omega\|_2^{\frac{1}{2}}\|\nabla\partial_x u_1\|_2^{\frac{1}{2}}\|\omega\|_2^{\frac{1}{2}}\|\omega_x\|_2^{\frac{1}{2}}\\ &\le \|\nabla\omega_x\|_2^2+C\|\Omega\|_2\|\nabla\partial_x u_1\|_2\|\omega\|_2\|\omega_x\|_2\\ &\le \|\nabla\omega_x\|_2^2+\|\nabla\partial_x u_1\|_2^2+C\|\Omega\|_2^2\|\omega_x\|_2^2\|\omega\|_2^2.\end{aligned}$$

Finally, we deal with J_6. There are two terms in J_6,

$$J_6=2\int\nabla\omega\cdot\nabla\Omega=-\int\omega_{xx}\Omega+2\int\omega_y\Omega_y.$$

They can be bounded as follows.

$$\begin{aligned}&\left|\int\omega_{xx}\Omega\right|\le\|\omega_{xx}\|_2\|\Omega\|_2\le\frac{1}{2}\|\nabla\omega_x\|_2^2+C\|\Omega\|_2^2,\\ &\int\omega_y\Omega_y=\int(\omega_y\partial_{xy}u_2-\omega_y\partial_{yy}u_1),\\ &\left|\int\omega_y\partial_{xy}u_2\right|\le C\|\nabla\omega\|_2\|\nabla\partial_y u_1\|_2,\\ &\left|\int\omega_y\partial_{yy}u_1\right|\le C\|\nabla\omega\|_2\|\nabla\partial_y u_1\|_2.\end{aligned}$$

Collecting the estimates above and applying Gronwall's inequality allows us to conclude the global H^1-bound. This completes the proof of Proposition 3.1. □

3.2 Global H^2 Bound and the proof of Theorem 1.2

This subsection proves Theorem 1.2. As we explained before, it suffices to prove the global H^2-bound. This is given in the following proof.

Proof of Theorem 1.2. Dotting (3.1) with $\nabla\Omega$, applying ∇ to (3.2) and then dotting with $\Delta\omega$, and dotting (3.3) with Δj, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2+\|\Delta\omega\|_2^2)+\|\Delta\partial_x u_1\|_2^2\\ &\qquad+\|\Delta\partial_y u_1\|_2^2+\|\nabla\partial_x j\|_2^2+2\|\Delta\omega_x\|_2^2+\|\Delta\omega\|_2^2\\ &=L_1+L_2+L_3+L_4+L_5+2L_6+L_7,\end{aligned}$$

where

$$L_1=-\int\nabla\Omega\cdot\nabla u\cdot\nabla\Omega\,dxdy,\qquad L_2=-\int\nabla j\cdot\nabla u\cdot\nabla j\,dxdy,$$
$$L_3=2\int\nabla\Omega\cdot\nabla b\cdot\nabla j\,dxdy,\qquad L_4=2\int\nabla[\partial_x b_1(\partial_x u_2+\partial_y u_1)]\cdot\nabla j\,dxdy,$$

$$L_5=-2\int\nabla[\partial_x u_1(\partial_x b_2+\partial_y b_1)]\cdot\nabla j\,dxdy,\qquad L_6=\int\Delta\Omega\Delta\omega\,dxdy,$$
$$L_7=-\int\Delta(u\cdot\nabla\omega)\Delta\omega\,dxdy.$$

We write the four terms in L_1 explicitly,

$$\begin{aligned}L_1&=-\int(\nabla\Omega\cdot\nabla u\cdot\nabla\Omega)\,dxdy\\&=-\int\big(\partial_x u_1(\partial_x\Omega)^2+\partial_x u_2\partial_x\Omega\partial_y\Omega+\partial_y u_1\partial_x\Omega\partial_y\Omega+\partial_y u_2(\partial_y\Omega)^2\big)\\&\equiv L_{11}+L_{12}+L_{13}+L_{14}.\end{aligned}$$

These terms can be bounded as follows.

$$\begin{aligned}L_{11}&\le C\|\partial_x\Omega\|_2^{\frac{3}{2}}\|\partial_x u_1\|_2^{\frac{1}{2}}\|\partial_{xx}u_1\|_2^{\frac{1}{2}}\|\partial_{xy}\Omega\|_2^{\frac{1}{2}}\\&\le C\|\nabla\Omega\|_2^{\frac{3}{2}}\|\partial_x u_1\|_2^{\frac{1}{2}}\|\nabla\partial_x u_1\|_2^{\frac{1}{2}}\|\Delta\partial_x u_1\|_2^{\frac{1}{2}}\\&\le\frac{1}{48}\|\Delta\partial_x u_1\|_2^2+C\|\partial_x u_1\|_2^{\frac{2}{3}}\|\nabla\partial_x u_1\|_2^{\frac{2}{3}}\|\nabla\Omega\|_2^2.\\L_{12}&=\int\partial_x u_2\partial_x\Omega\partial_y\Omega\\&\le C\|\partial_x u_2\|_2\|\partial_x\Omega\|_2^{\frac{1}{2}}\|\partial_{xy}\Omega\|_2^{\frac{1}{2}}\|\partial_y\Omega\|_2^{\frac{1}{2}}\|\partial_{xy}\Omega\|_2^{\frac{1}{2}}\\&\le C\|\partial_x u_2\|_2\|\nabla\Omega\|_2\|\partial_{xy}\Omega\|_2\\&\le\frac{1}{48}\|\Delta\partial_x u_1\|_2^2+C\|\partial_x u_2\|_2^2\|\nabla\Omega\|_2^2.\end{aligned}$$

We note that $\partial_{xy}\Omega=\partial_{xy}(\partial_x u_2-\partial_y u_1)=(-\partial_{xxx}-\partial_{xyy})u_1$, and thus

$$\|\partial_{xy}\Omega\|_2^2\le C\|\Delta\partial_x u_1\|_2^2.$$

Therefore,

$$\begin{aligned}L_{13}&\le\frac{1}{48}\|\partial_{yy}\Omega\|_2^2+C(\|\partial_y u_1\|_2^2+\|\partial_{xy}u_1\|_2^2)\|\nabla\Omega\|_2^2.\\L_{14}&\le\frac{1}{48}\|\partial_{yy}\Omega\|_2^2+C\|u_2\|_2^2\|\Omega\|_2^2\|\nabla\Omega\|_2^2.\end{aligned}$$

The rest of the terms are bounded as follows.

$$\begin{aligned}L_2&\le\frac{1}{48}\|\nabla\partial_x j\|_2^2+C\|\partial_{xy}u_1\|_2^{\frac{2}{3}}\|\Omega\|_2^{\frac{2}{3}}\|\nabla j\|_2^2+\|\Omega\|_2^2\|\nabla j\|_2^2.\\L_3&\le\frac{1}{48}\|\nabla\partial_x j\|_2^2+C(\|j\|_2^2+\|\partial_x j\|_2^2)(\|\nabla\Omega\|_2^2+\|\nabla j\|_2^2).\\L_4&\le\frac{1}{48}\|\nabla\partial_x j\|_2^2+\frac{1}{48}\|\Omega_{yy}\|_2^2+C(\|\Omega\|_2^2+\|j\|_2^2+\|\partial_x j\|_2^2)(\|\nabla\omega\|_2^2+\|\nabla j\|_2^2).\end{aligned}$$

L_5 admits a similar bound as L_4.

$$L_6 \le \frac{1}{48}\|\Delta\omega_x\|_2^2 + \frac{1}{48}\|\Delta\partial_y u_1\|_2^2 + C(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2$$
$$L_7 \le \frac{1}{48}\|\Delta\omega_x\|_2^2 + \frac{1}{48}\|\Delta\partial_y u_1\|_2^2 + C(\|\nabla\omega\|_2^2 + \|\nabla\omega_x\|_2^2 + \|\Omega\|_2^2)(\|\nabla\Omega\|_2^2 + \|\Delta\omega\|_2^2).$$

Combining the bounds above and then applying Gronwall's inequality lead to the desired global H^2-bound. This completes the proof of Theorem 1.2. □

4 Proof of Theorem 1.3

This section proves Theorem 1.3. Due to the similarity to the proofs in the previous two sections, we shall omit most of the details and provide only the estimates that are significantly different from the previous ones.

First, the following global L^2-bound holds.

Lemma 4.1. *Assume that (u_0,b_0,ω_0) satisfies the conditions in Theorem* 1.3. *Let (u,b,ω) be the corresponding solution of* (1.6). *Then, (u,b,ω) obeys the following global L^2-bound,*

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \|\omega(t)\|_{L^2}^2 \\ &\qquad + 2\int_0^t \|\partial_x u_1, \partial_y u_2\|_2^2 d\tau + 2\int_0^t \|\partial_x b(\tau)\|_{L^2}^2 d\tau + 2\int_0^t \|\partial_x \omega(\tau)\|_{L^2}^2 d\tau \\ &\le C(\|u_0\|_{L^2}^2, \|b_0\|_{L^2}^2, \|\omega_0\|_2^2)\end{aligned}$$

for any $t \ge 0$.

The following global H^1-bound can also be established.

Proposition 4.1. Assume that (u_0,b_0,ω_0) satisfies the conditions in Theorem 1.3. Let (u,b,ω) be the corresponding solution of (1.6). Then (u,b,ω) satisfies, for any $T>0$,

$$u,b,\omega \in C([0,T];H^1).$$

We briefly indicate the proof of Proposition 4.1. Again we invoke the equations of $\Omega = \nabla\times u$, $\nabla\omega$ and $j = \nabla\times b$

$$\begin{cases}\Omega_t + u\cdot\nabla\Omega = -\Delta\partial_y u_1 + (b\cdot\nabla)j - \Delta\omega, \\ \partial_t\nabla\omega + \nabla(u\cdot\omega) + 2\nabla\omega = \nabla\omega_{xx} + \nabla\Omega, \\ j_t + u\cdot\nabla j = \partial_{xx} j + b\cdot\nabla\Omega + 2\partial_x b_1(\partial_x u_2 + \partial_y u_1) - 2\partial_x u_1(\partial_x b_2 + \partial_y b_1), \\ \nabla\cdot u = 0, \quad \nabla\cdot b = 0.\end{cases}$$

Dotting the equations above with Ω, $\nabla\omega$ and j, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2 + \|j\|_{L^2}^2 + \|\nabla\omega\|_2^2\right) + \|(\nabla\partial_x u_1, \nabla\partial_y u_1)\|_2^2$$

$$
\begin{aligned}
&\quad +\|\partial_x j\|_{L^2}^2+\|\nabla\omega_x\|_2^2+2\|\nabla\omega\|_2^2\\
&=2\int\left[\partial_x b_1(\partial_x u_2+\partial_y u_1)j-\partial_x u_1(\partial_x b_2+\partial_y b_1)\right]jdxdy\\
&\quad -\int\nabla\omega\cdot\nabla u\cdot\nabla\omega+2\int\nabla\omega\cdot\nabla\Omega\\
&\equiv J_1+J_2++J_3+J_4+J_5+J_6.
\end{aligned}
$$

All these term can be bounded as in the previous cases except the term $\int\partial_x u_2\omega_x\omega_y$, which can be handled as follows. We first integrate by parts to obtain

$$\int\partial_x u_2\omega_x\omega_y=-\int u_2\partial_{xx}\omega\omega_y-\int u_2\omega_x\partial_{xy}\omega.$$

The two terms on the right are bounded by

$$
\begin{aligned}
\left|-\int u_2\partial_{xx}\omega\omega_y\right| &\le C\|\partial_{xx}\omega\|_2\|u_2\|_2^{\frac12}\|\partial_y u_2\|_2^{\frac12}\|\omega_y\|_2^{\frac12}\|\partial_{xy}\omega\|_2^{\frac12}\\
&\le C\|\nabla\partial_x\omega\|_2^{\frac32}\|u_2\|_2^{\frac12}\|\partial_y u_2\|_2^{\frac12}\|\nabla\omega\|_2^{\frac12}.\\
&\le\|\nabla\omega_x\|_2^2+C\|u_2\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2\\
\left|-\int u_2\omega_x\partial_{xy}\omega\right| &\le C\|\partial_{xy}\omega\|_2\|u_2\|_2^{\frac12}\|\partial_y u_2\|_2^{\frac12}\|\omega_x\|_2^{\frac12}\|\partial_{xx}\omega\|_2^{\frac12}\\
&\le\|\nabla\omega_x\|_2^2+C\|u_2\|_2^2\|\partial_y u_1\|_2^2\|\nabla\omega\|_2^2.
\end{aligned}
$$

Collecting the estimates would yield the desired global H^1-bound.

As in the previous two cases, we can prove Theorem 1.3 by establishing the global H^2-bound. Since the process of proving the global H^2-bound is similar to the previous two cases, we shall omit the details.

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References

[1] G. Ahmadi, M. Shahinpoor, *Universal stability of magneto-micropolar fluid motions*, Int. J. Engng. Sci. **12** (1974), 657-663.

[2] C. Cao, D. Regmi, J. Wu, *The 2D MHD equations with horizontal dissipation and horizontal magnetic diffusion*, J. Differential Equations, **254** (2013), 2661-2681.

[3] C. Cao, J. Wu, *Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion*, Adv. in Math. **226** (2011), 1803-1822.

[4] C. Cao, J. Wu, B. Yuan, *The 2D incompressible magnetohydrodynamics equations with only magnetic diffusion*, SIAM J. Math. Anal. **46** (2014), 588-602.

[5] J. Cheng, Y. Liu, *Global regularity of the 2D magnetic micropolar fluid flows with mixed partial viscosity*, Computer and Mathematics with Applications, **70**(2015), 66-72.

[6] S. C. Cowin, *Polar fluids*, Phys. Fluids, **11** (1968), 1919-1927.

[7] B. Dong, Z. Zhang, *Global regularity of the 2D micropolar fluid flows with zero angular viscosity*, J. Differential Equations, **249**(2010) 200-213.

[8] M. E. Erdogan, *Polar effects in the apparent viscosity of suspension*, Rheol. Acta **9** (1970), 434-438.

[9] A. C. Eringen, *Theory of micropolar fluids*, J. Math. Mech. **16** (1966), 1-18.

[10] A. C. Eringen, *Micropolar fluids with stretch*, Int. J. Engng. Eci. **7** (1969), 115-127.

[11] J. Fan, H. Malaikah, S. Monaquel, and *et al.*, *Global Cauchy problem of 2D generalized MHD equations*, Monatsh Math, **175**(2014), 127-131.

[12] X. Hu, F. Lin, *Global existence for two dimensional incompressible magnetohydrodynamic flows with zero magnetic diffusivity*, arXiv: 1405.0082v1 [math.AP] 1 May 2014.

[13] Q. Jiu, J. Zhao, *Global regularity of 2D generalized MHD equations with magnetic diffusion*, Z. Angew. Math. Phys. **66** (2015), 677-687.

[14] Z. Lei, *On Axially Symmetric incompressible magnetohydrodynamics in three dimensions*, J. Differential Equations, **259** (2015), 3202-3215.

[15] Z. Lei, Y. Zhou, *BKM's criterion and global weak solutions for magnetohydrodynamics with zero viscosity*, Discrete Contin. Dyn. Syst. **25** (2009), 575-583.

[16] F. Lin, L. Xu, P. Zhang, *Global small solutions to 2-D incompressible MHD system*, J. Differential Equations, **259** (2015), 5440C5485.

[17] G. Lukaszewick, *Micropolar Fluids: Theory and Applications*, Birkhauser, Boston, 1999.

[18] E. E. Ortega-Torres, M. A. Rojas-Medar, *Magneto-micropolar fluid motion: global existence of strong solutions*, Abstr. Appl. Anal. **4**(1999), 109-125.

[19] D. Regmi, *Global weak solutions for the two-dimensional magnetohydrodynamic equations with partial dissipation and diffusion*, preprint.

[20] X. Ren, J. Wu, Z. Xiang, Z. Zhang, *Global existence and decay of smooth solution for the 2-D MHD equations without magnetic diffusion*, J. Functional Analysis **267** (2014), 503-541.

[21] M. A. Rojas-Medar, *Magneto-micropolar uid motion: existence and uniqueness of strong solutions*, Math. Nachr. **188** (1997), 301-319.

[22] C. Trann, X. Yu, Z. Zhai, *On global regularity of 2D generalized magnetohydrodynamic equations*, J. Differential Equations, **254** (2013), 4194-4216.

[23] J. Wu, *Generalized MHD equations*, J. Differential Equations, **195** (2003), 284-312.

[24] J. Wu, *Regularity criteria for the generalized MHD equations*, Comm. Partial Differential Equations, **33** (2008), 285-306.

[25] J. Wu, *Global regularity for a class of generalized magnetohydrodynamic equations*, J. Math. Fluid Mech. **13** (2011), 295-305.

[26] J. Wu, Y. Wu, X. Xu, *Global small solution to the 2D MHD system with a velocity damping term*, SIAM J. Math. Anal. **47** (2015), 2630-2656.

[27] K. Yamazaki, *Global regularity of logarithmically supercritical MHD system with zero diffusivity*, Appl. Math. Lett. **29**(2014), 46-51.

[28] K. Yamazaki, *On the global regularity of two-dimensional generalized magnetohydrodynamics system*, J. Math. Anal. Appl. **416** (2014), 99-111.

[29] K. Yamazaki, *Global regularity of the two-dimensional magneto-micropolar fluid system with zero angular viscosity*, Discrete Contin. Dyn. Syst. **35** (2015), 2193-2207.

[30] B. Yuan, *Regularity of weak solutions to magneto-micropolar uid equations*, Acta Math. Sci. Ser. B Engl. Ed. **30B** (2010), 1469-1480.

[31] T. Zhang, *An elementary proof of the global existence and uniqueness theorem to 2-D incompressible non-resistive MHD system*, arXiv:1404.5681v1 [math.AP] 23 Apr 2014.