My research interests are *nonlinear partial differential equations and numerical methods.* My works are focused on the dynamics of single and multi phase nonlinear fluid flows in porous media, and finite element methods for equations arising from physics and engineering.

1. OVERVIEW

Generalized Forchheimer equations. The generalized Forchheimer equations are introduced to describe more complex regimes of fluid flows in porous media when Darcy's law becomes inadequate. Although widely acknowledged in engineering, the Forchheimer equations still do not have as much mathematical analysis as Darcy's law. This is partly due to the difficulty lying in their genuine nonlinearity. In my studies with collaborators, we use sophisticated methods as well as create the new ones to analyze and bring new understanding to these important flows.

For single-phase flows, we investigate generalized Forchheimer equations for slightly compressible fluids in porous media subject to the non-homogeneous time-dependent boundary conditions. The system describing its dynamics is reduced to a degenerate parabolic equation for the pressure. Previous works on the model are mainly numerical, or focused on the L^2-therory under the so-called Degree Condition (DC). My first contribution, together with co-authors, is to develop the L^∞-theory mainly in the (DC) case which is condition on degree of Forchheimer polynpmial. The second contribution is to develop the L^α-theory for $\alpha \in (0,\infty)$ that removes the (DC) restriction. In both developments, we derive various explicit estimates for solutions in terms of the data in Lebesgue and Sobolev norms, particularly for large time. These result in the establishment of long-time asymptotic behavior of solutions and their derivatives, and the structural stability with respect to the boundary data, even when the data are unbounded at time infinity. Furthermore, we establish the quantified continuous dependence of the solutions on the coefficients of the Forchheimer polynomials. This is particularly important in practice since these coefficients are determined by matching the field data. To obtain the above results we utilize the special structure of the equation, particularly the monotonicity and its perturbation form. Variations of techniques by De Giorgi, Ladyzhenskaya, Uraltseva and Di Benedetto are used to study local properties of the solutions. For long-time dynamics we combine these with the nonlinear Gronwall and uniformly Gronwall-type inequalities to obtain the stability results. The methods developed in our works are applicable to other degenerate parabolic equations of similar structures.

For multi-phase flows, L. Hoang, A. Ibragimov and I study the generalized Forchheimer equations for two incompressible, immiscible fluids in porous media with the presence of the capillary pressure. The flows are described by a complicated system of nonlinear partial differential equations. Its complexity limits the current analysis of the flows' dynamics. We find a way to rewrite the system in a simpler form that, more importantly, allows us to study its dynamical properties. Using this system, we obtain all steady states in the one-dimensional case, or a family of non-constant ones in the multi-dimensional case with certain invariance. Their existence and properties are showed to depend on the relations between the capillary pressure, each phase's relative permeability and Forchheimer polynomials. To analyze those steady states' stability, we derive the linearized system and reduce it further to a parabolic equation for the saturation. This equation has a special structure depending on the steady states which we discover and exploit to establish various stability results in both bounded and unbounded domains. Our specific techniques are the two new forms of the lemma of growth of Landis-type, and also Bernstein's *a priori* estimates.

Relevant references: [19, 20, 21] on single-phase flows, and [17, 18] on multi-phase flows.

Numerical methods. R. Kirby and I apply the Galerkin finite element method to a class of nonlinear Klein-Gordon equations. Equations of this form appear frequently in geometry and many areas of physics, including relativistic quantum mechanics and superconductors. We give the optimal-order error estimates of Galerkin finite element methods for non-Lipschitz nonlinearity. The result holds in dimensions one and two for general nonlinear term with the minimal sign condition, and in dimension three under a certain growth condition. Our results are the first known for the non-Lipschitz nonlinearity. Time-stepping schemes and numerical results are analyzed to strengthen the theoretical result.

Next, we investigate the mixed finite element spaces in discretization of acoustic wave equations. These equations are of essential interest in many applications such as seismic imaging. We apply mixed finite element approximations to the first-order form of the acoustic wave equation. Our semidiscrete method exactly conserves the system energy. Furthermore, we show that with a symplectic Euler time discretization, a perturbed energy quantity exactly conserves perturbed energy that is positive-definite and equivalent to the actual energy under a CFL condition (see Fig. 1). In addition to proving optimal-order $L^\infty(L^2)$ estimates, we derive stability and error bounds for the time derivatives and divergence of solutions which go beyond the standard estimates in literature. The proof is based on a new bootstrap technique.

On the subject of efficient implementations of numerical schemes, R. Kirby and I use the Bernstein bases in finite element method to develop fast algorithms to compute the mass and stiffness matrices. In order to develop the fast simplicial quadrature-based finite element operators, we derive low-complexity matrix-free finite element algorithms for simplicial Bernstein polynomials on simplices. The techniques make use of a sparse representation of differentiation and special block structure in the matrices by evaluating the B-form polynomials at warped Gauss points. These algorithms can be applied to problems with both constant and variable coefficients (see [22]).

Relevant references: [24, 23, 22].

2. GENERALIZED FORCHHEIMER FLOWS IN POROUS MEDIA

The nonlinear Forchheimer equations are considered as laws of hydrodynamics in porous media in case of high Reynolds numbers, when the fluid flows deviate from the Darcy's law (c.f. [3, 29]). The original two term, three term, and power laws are extended to the generalized Forchheimer equations [2]:

$$g(|u|)u = -\nabla p, \tag{1}$$

where u is the velocity, p is the pressure, and

$$g(s) = a_0 s^{\alpha_0} + a_1 s^{\alpha_1} + \dots + a_N s^{\alpha_N}, s \geq 0, \tag{2}$$

with $N > 0, a_0, a_N > 0, a_1, \dots, a_{N-1} \geq 0, \alpha_0 = 0 < \alpha_1 < \dots < \alpha_N$. Here, the function $g(s)$ is called the Forchheimer polynomial. The generalized Forchheimer equation can be inverted to

$$u = -K(|\nabla p|)\nabla p$$

with conductivity function K degenerating for large ∇p. For slightly compressible fluids, the description of fluid dynamics can be deduced, with a slight simplification, to a degenerate parabolic equation for the pressure $p(x,t)$:

$$\frac{\partial p}{\partial t} = \nabla \cdot (K(|\nabla p|)\nabla p). \tag{3}$$

My collaborators and I study the initial boundary value problem (IBVP) for this equation in an open bounded domain U in $\mathbb{R}^n$ with Dirichlet boundary data. In [2, 15, 16], the authors establish the continuous dependence of the solutions of (3) in L^2-norm and $W^{1,2-a}$-norm, where $a = \frac{\deg g}{\deg g+1} \in (0,1)$, on the initial, boundary data and on the Forchheimer polynomials. To study the asymptotic dynamics of the solutions, they introduced a technical Degree Condition (DC), namely, $\deg(g) \leq \frac{n}{n-2}$. This condition also arises naturally in studies of degenerate parabolic equations. (See e.g. [9, 11] for the cruciality of such condition in establishing Harnack inequalities.) We establish a L^α-theory for $\alpha \neq 2$, as a counterpart for the L^2-results in [15], and to explore the problem when the (DC) is not met which is referred as (NDC). In contrast to related long-time dynamics results in the L^2 in [15] which require (DC) condition, the results obtained for L^α-spaces are generally without any restriction on degree of polynomial. For this study, new Poincaré-Sobolev inequalities, monotonicity, and nonlinear Gronwall-type estimates for nonlinear differential inequalities are utilized to achieve better asymptotic bounds. The methods developed are general and can be applied to other degenerate parabolic equations of similar structure.

Due to the absence of maximum principle, the L^α-theory are essential to the L^∞-theory for the higher regularity spaces. We study the pressure and its time derivative in space L^∞, the pressure gradient in L^s for any $1 \leq s < \infty$ and the pressure Hessian in $L^{2-\delta}$ for $\delta \in (0,a]$. Our high priority is the long-time dynamical properties, including uniform estimates in time, asymptotic bounds and asymptotic stability. Such topics of long-time dynamics of degenerate parabolic equation, particularly in L^∞, is important, and the specific results are usually hard to obtain. (See, for e.g., [33, 30].) To deal with this, we combine iteration techniques by De Giorgi [8] and Ladyzhenskaya-Uraltseva [27], which were primarily used for studying local properties of solutions to elliptic and parabolic problems, with those from long-time dynamics studies for nonlinear partial differential equations such as Navier-Stokes equations [12]. For our degenerate equations, we also use and refine relevant techniques in DiBenedetto's book [9]. Such a combination gives fruitful results on the estimates of solutions for large time as well as detailed continuous dependence of the solutions on time-dependent boundary data and coefficients of the Forchheimer polynomials. We also emphasize that the mentioned general techniques from parabolic equations must be used in accordance with the structure of my equation, in this case, the important monotonicity and perturbed monotonicity.

The main features of the results for single-phase flows include:

- L^{α}-estimates of the solutions with any $\alpha \in (0, \infty]$ for all time, large time and time infinity.
- The L^{2-a}-estimates up to the boundary of gradient of pressure and interior L^s-estimates for any $s \geq 1$ of gradient of pressure.
- The L^2-estimates on whole domain and interior L^{∞}-estimates of time derivative of solutions.
- The interior $L^{2-\delta}$-estimates for Hessian $\nabla^2 p$.
- The continuous dependence of the solution in L^{α}-norm for $\alpha \in (0, \infty)$ and the pressure gradient in L^{2-a}-norm on the boundary data and the Forchheimer polynomial. Futhermore these are established for solutions in interior L^{∞}-norms and for gradient of solutions in interior $L^{2-\delta}$-norms for finite time intervals, and for time $t \to \infty$. The obtained results show that even when each individual data ψ_1, ψ_2 grows unbounded as time $t \to \infty$, the difference between two corresponding solutions p_1, p_2 can be small provided the difference $\psi_1 - \psi_2$ is small. Similarly, the smallness of the difference between two solutions corresponding to two Forchheimer equations, when $t \to \infty$, can be controlled by the difference between coefficient vectors of the two Forchheimer polynomials.

3. TWO-PHASE GENERALIZED FORCHHEIMER FLOWS

L. Hoang, A. Ibragimov and I study n-dimensional two phase flows in porous media with constant porosity ϕ between 0 and 1. Each position $\mathbf{x} \in \mathbb{R}^n$ in the medium is considered to be occupied by two fluids called phase 1 (for example, water) and phase 2 (for example, oil). Saturation, density, velocity, and pressure for each ith-phase ($i = 1, 2$) are $S_i \in [0, 1]$, $\rho_i \geq 0$, $u_i \in \mathbb{R}^n$, and $p_i \in \mathbb{R}$, respectively. The saturation functions naturally satisfy

$$S_1 + S_2 = 1. \tag{4}$$

Each phase's velocity is assumed to obey the generalized Forcheimer equation:

$$g_i(|u_i|)u_i = -\tilde{f}_i(S_i)\nabla p_i, \quad i = 1, 2, \tag{5}$$

where $\tilde{f}_i(S_i)$ is the relative permeability for the ith phase, and each g_i is Forchheimer polynomial defined as in (2). Conservation of mass commonly holds for each of the phases:

$$\partial_t(\phi \rho_i S_i) + div(\rho_i u_i) = 0, \quad i = 1, 2. \tag{6}$$

Due to incompressibility of the phases, i.e. $\rho_i = const. > 0$, Eq. (6) is reduced to

$$\phi \partial_t S_i + div\, u_i = 0, \quad i = 1, 2. \tag{7}$$

In the presence of the capillary pressure p_c, we have the relation

$$p_c = p_1 - p_2. \tag{8}$$

Current analysis of two-phase Darcy flows in literature is mainly focused on the existence of weak solutions [6, 4] and their regularity [26, 10]. However, questions about the stability and dynamics are not answered. The nonlinearity of the relative permeabilities and capillary pressure and their imprecise characteristics near the extreme values make it hard to analyze the modeling PDE system. The two-phase generalized Forchheimer flows (4)–(8) are even more difficult due to the additional nonlinearity in the momentum equation. For example, unlike the Darcy flows, there is no Kruzkov-Sukorjanski transformation [25] to convert the system to a convenient form for the total velocity. Therefore, we approach this system differently than the previous works.

First, we rewrite the whole system in a simpler form which allows us to study its dynamics. Denote $S = S_1$, the relative permeabilities and capillary pressure are re-denoted as functions of S, that is, $\tilde{f}_1(S_1) = f_1(S)$, $\tilde{f}_2(S_2) = f_2(S)$ and $p_c = p_c(S)$. Define $G_i(u) = g_i(|u|)u$, $u \in \mathbb{R}^n$ and $F_i(S) = (p_c'(S) f_i(S))^{-1}, i = 1, 2$. From the equations (4), (5) and (7), we derive a nonlinear partial differential equations (PDE) system for the unknowns $S = S(x, t)$, $u_1 = u_1(x, t)$ and $u_2 = u_2(x, t)$:

$$0 \leq S = S(x, t) \leq 1, \tag{9a}$$
$$S_t = -div\, u_1, \tag{9b}$$
$$S_t = div\, u_2, \tag{9c}$$
$$\nabla S = F_2(S)G_2(u_2) - F_1(S)G_1(u_1). \tag{9d}$$

We study the system (9) for one dimensional and axially symmetric flows in the higher dimensional space. The existence of steady states are proved. Basic (and physically relevant) assumptions are made on the capillary pressure and relative permeabilities which enable us to prove the existence of non-constant steady states in $(0,1)$. For stability study, we linearize system(9) at these steady states, deduce from this linearized system to parabolic equation for saturation, and convert it to a convenient form for the study of sup-norm of solutions. The coefficients of this equation is a function derived from saturation, permeability and capillary pressure.

For 1-dimensional case, we establish the weighted stability for the perturbation and also derive long time estimates for its weighted L^∞-norm. The stability for velocities (on bounded intervals) is obtained by using Bernstein's estimate technique.

For n-dimesional case we prove on the bounded domain the asymptotic stability results by utilizing a variation of Landis's lemma of growth in time variable. The Bernstein's a priori estimate technique is used in proving interior continuous dependence of the velocities on the initial and boundary data. On the unbounded domain, the maximum principle is proved and used to obtain the stability of the zero solution. We also prove a lemma of growth in the spatial variables by constructing particular barriers (super solutions) using the specific structure of the linearized equation for saturation. Using this, we prove a dichotomy theorem on the solution's behavior, and ultimately show that the solution, on any finite time interval, decays to zero as $|x| \to \infty$. For time tending to infinity, we find an increasing, continuous function $r(t) > 0$ with $r(t) \to \infty$ as $t \to \infty$ such that along any curve $x(t)$ with $|x(t)| \geq r(t)$, the solution goes to zero.

4. SYMPLECTIC-MIXED FINITE ELEMENT APPROXIMATION OF LINEAR WAVE EQUATIONS

R. Kirby and I consider the linear acoustic wave equation

$$\begin{aligned} \varrho p_t + \nabla \cdot u &= f, \\ \kappa^{-1} u_t + \nabla p &= g, \end{aligned} \tag{10}$$

posed on some domain $\Omega \times [0,T] \subset \mathbb{R}^d \times \mathbb{R}$ with $d = 2, 3$. Assume T is finite and, for simplicity, that Ω is polyhedral so that it may be tesselated exactly into simplices. We pose initial conditions $p(\cdot,0) = p_0(\cdot)$ and $u(\cdot,0) = u_0(\cdot)$ and the boundary condition $u \cdot \nu = 0$ on $\partial\Omega$, where ν is the unit outward normal to Ω. Assume the material density, ϱ, is some measurable function bounded below and above by positive numbers ϱ_* and ϱ^*. The parameter κ is the bulk modulus of compressibility, assumed bounded between positive numbers κ_* and κ^*. Geveci [13] first applied $H(\mathrm{div})$ finite elements to a wave equation, proving existence and uniqueness and optimal *a priori* error estimates in $L^\infty(L^2)$ to both variables p, u for the formulation (10). He also formulated but did not analyze a backward Euler time-stepping scheme.

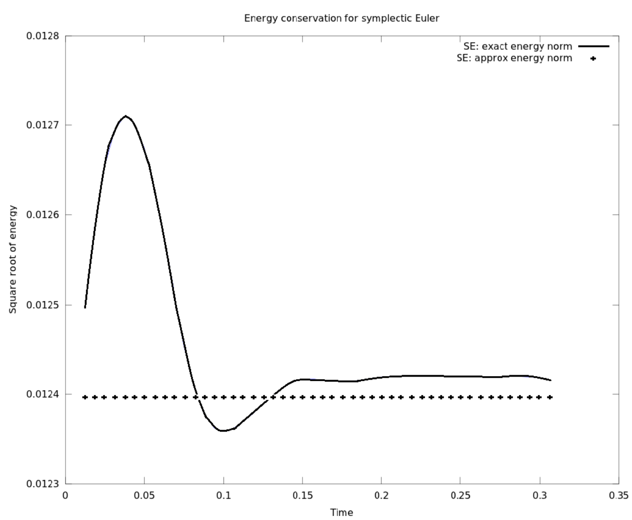

Figure 1: Conservation of energy.

Our work strengthens the existing theory in two major ways. First, we are able to control the temporal derivatives of variables p, u and the divergence of u; such estimates are new. Second, Geveci uses Hamiltonian structure in his semidiscrete analysis. We carry this consideration forward in our discussion of fully discrete methods, considering the energy conservation properties of the symplectic Euler method [5]. In this way, our discretization preserves the essential structure in both the spatial and temporal aspects of the wave equation. This combination has been formulated for electromagnetics [31], but ours is the first theoretical analysis combining symplectic time integration with some form of mixed finite element space. The main results are summarized as follows:

- We apply mixed finite element approximations to the first-order form of the acoustic wave equation. The semidiscrete method exactly conserves the system energy.
- We show that with a symplectic Euler time discretization, the method exactly conserves a perturbed energy quantity that is positive-definite and equivalent to the actual energy under a CFL condition (see Fig. 1).
- In addition to proving optimal-order $L^\infty(L^2)$ estimates, We also develop a bootstrap technique that enables me to derive stability and error bounds for the time derivatives and divergence of u.

5. GALERKIN FINITE ELEMENT METHODS FOR NONLINEAR KLEIN-GORDON EQUATIONS

We consider the second-order nonlinear hyperbolic equation for the unknown $u = u(x,t)$

$$u_{tt} - \nabla \cdot (a(u)\nabla u) + f(u) = g \text{ on } \Omega \times (0,T] \tag{11}$$

with homogeneous Dirichlet boundary and the initial conditions $u(\cdot,0) = u_0(\cdot), u_t(\cdot,0) = v_0(\cdot)$. The bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2,3$ has boundary $\partial\Omega \in C^2$. Assume T is positive and finite. The function $a(\cdot) \in C^2(\mathbb{R})$, $a(u) > C > 0$ for all $u \in \mathbb{R}$, $a(\cdot)$ and its derivative are Lipschitz continuous functions. The function $f \in C^1(\mathbb{R})$ satisfies

$$zf(z) \geq 0 \ \ \forall z \in \mathbb{R}, \ \ f(0) = 0 \quad \text{and} \quad |f'(u)| \leq C(1+|u|^p), \ \ \text{where } 0 \leq p \leq \frac{2}{d-2}. \tag{12}$$

We study the stability and superconvergence of both semidiscrete and fully discrete schemes for equation (11) under assumption (12) using Galerkin finite element methods. The superconvergence in the Lipschitz case and optimal rates in the non-Lipschitz case are established. The proof of convergence in the case of non-Lipschitz nonlinearites relies on energy estimates combined with embedding theorems. Our techniques work in dimension two for any strength of nonlinearity so that we cover similar results as in [14]. Though restricted in dimension three to $p \leq 2$, same as Lions's results in [28], our result is the first known convergence proof of a finite element method in the non-Lipschitz case. We also analyze energy conservation on some of time-stepping schemes. The numerical results confirm the theoretical predictions and show that leapfrog time-stepping conserves energy reasonably well.

6. ON GOING RESEARCH AND LONG-TERM RESEARCH

This section contains brief a description of my current research projects and future research.

1. I establish the L^∞-estimate for solutions in Sobolev norm to generalized Forchheimer equations for slightly compressible fluids in porous media subject to the boundary Dirichlet condition. Also the time derivative and Hessian of solutions in L^∞-estimate are obtained. No degree conditions are imposed in this study.
2. My current focus is qualitative properties of solutions to Forchheimer-Ward equations for slightly compressible fluids and generalized Forchheimer flows with non-homogeneous boundary condition for pressure in porous media. The structural stability in L^∞-norm are also studied.
3. I establish velocity-pressure formulation to two-phase Forchheimer equations for incompressible-slightly compressible fluids. Prove the existence of steady states and their linear stability.
4. My interests are the error-estimate, convergence of finite element Galerkin approximation to solutions and their derivatives for nonlinear hyperbolic equations and generalized Forchheirmer equations.
5. I am interested in studying the affects of soft errors as well as developing new fault-resilient algorithms for scientific simulations.
6. My study about the spectra of Bernstein-based FEM operators, fast algorithms to find mass and stiffness matrix using the B-form of polynomials over d-dimensional simplices. These rely on special properties of the Bernstein basis.

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