Unit 8 Inverse Trig & Polar Form of Complex Nums.

(2) Inverse Functions
(3) Invertibility of Trigonometric Functions
(4) Inverse Sine and Inverse Cosine
(6) Inverse Tangent
(7) Other Inverse Trig Functions
(12) Review of Complex Numbers
(13) Graphical Representation of a Complex Number
Absolute Value of a Complex Number
(14) Polar Form of a Complex Number
(16) Polar Coordinates
(17) Multiplying and Dividing Complex Numbers
DeMoivre's Theorem

This is a SCIENTIFIC OR GRAPHING CALCULATORS ALLOWED unit.

Know the meanings and uses of these terms:

Modulus of a complex number
Polar coordinates

Review the meanings and uses of these terms:

One-to-one function
Inverse function
Complex number

Inverse Functions

Recall that an inverse operation mathematically "undoes" another operation.

For example, addition and subtraction are inverse operations while multiplication and division are inverse operations (except if dividing by 0).

Recall from algebra the definition of an inverse function:

Definition: An inverse function is a one-to-one function such that if a one-to-one function *f* with domain *A* and range *B* exists, then the inverse function f^{-1} (read as "*f* inverse") has domain *B* and range *A* and the following property is satisfied:

$$f(x) = y \Leftrightarrow f^{-1}(y) = x$$

for all *x* in *A* & all *y* in B.

(Further recall that a function *f* is said to be one-to-one if for every value *f*(*x*) in the range of *f* there is exactly one corresponding *x*-value in the domain of *x*. Any function which is not one-to-over its entire domain can be made one-to-one if its domain is appropriately limited.)

Invertibility of Trigonometric Functions

Trigonometric functions, such as sine, cannot have an inverse function over their entire domain because they are not one-to-one over their entire domain.

For example, $\sin\frac{\pi}{3} = \sin\frac{2\pi}{3} = \sin\frac{7\pi}{3} = \frac{\sqrt{3}}{2}$.

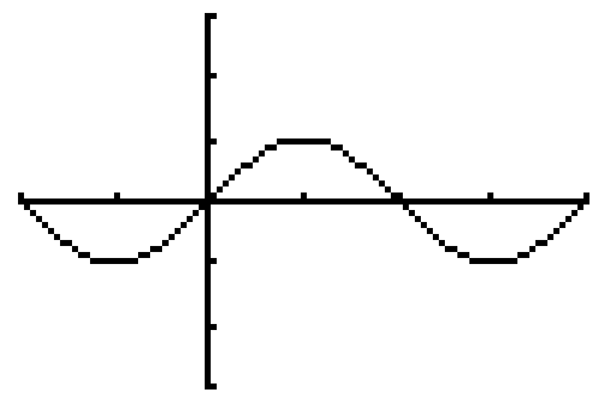
So the question becomes, if we want an inverse for a trigonometric function such as sine, what limitations must be placed upon the domain?

Defining trigonometric functions to be one-to-one

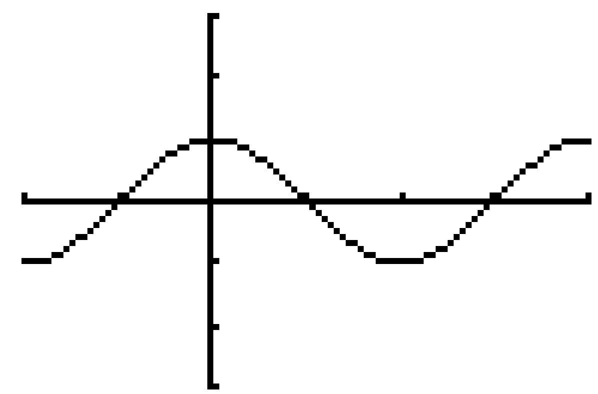
When making a function one-to-one for the purposes of defining an inverse function, it is important that the range is unchanged.

Sine and cosine both have a range of [-1, 1].

The questions is what is the simplest interval over which sine and cosine maintains a range of [-1, 1].



$y = \sin x$



$y = \cos x$

Inverse sine

Definition: The inverse sine function is the function $\sin^{-1}$ with domain [-1, 1] and range [-π/2, π/2] defined by

$$\sin^{-1} x = y \quad \Leftrightarrow \quad \sin y = x$$

Note: Another name for "inverse sine" is "arcsine" denoted as **arcsin**.

Based on our knowledge of inverse functions, we know that the inverse sine of *x* is the number between -π/2 and π/2 whose sine is *x*.

Further, $\sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$,

and $\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

Inverse cosine

Definition: The inverse cosine function is the function $\cos^{-1}$ with domain [-1, 1] and range [0, π] defined by

$$\cos^{-1} x = y \quad \Leftrightarrow \quad \cos y = x$$

Note: Another name for "inverse cosine" is "arccosine" denoted as **arccos**.

Thus, similar to the behavior of sine and inverse sine, we know that the inverse cosine of *x* is the number between 0 and π whose cosine is *x*.

Further, $\cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$,

and $\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$.

Find the exact value of each expression, if it is defined.

Ex. 1: $\sin^{-1}(0)$ $\cos^{-1}(0)$

Ex. 2: $\sin^{-1}(-1)$ $\cos^{-1}(-1)$

Ex. 3: $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Ex. 4: $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $\cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

Find the exact value of each expression, if it is defined.

Ex. 1: $\cos\left(\cos^{-1}\frac{3}{8}\right)$

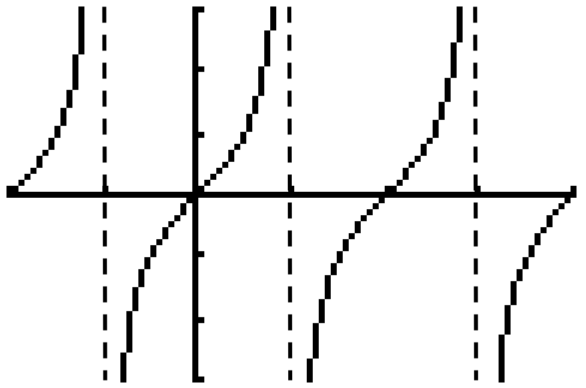
Ex. 2: $\sin\left(\sin^{-1}2\right)$

Ex. 3: $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$

Ex. 4: $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

Inverse tangent

Since tangent has a range of (-∞, ∞), the interval to limit the domain of tangent can be between two asymptotes.



Definition: The inverse tangent function is the function **tan^{-1}** with domain (-∞, ∞) and range (-π/2, π/2) defined by

$$\tan^{-1} x = y \quad \Leftrightarrow \quad \tan y = x$$

Note: Another name for "inverse tangent" is "arctangent" denoted as **arctan**.

And so, $\tan(\tan^{-1} x) = x$ for $x \in \mathbb{R}$,

and $\tan^{-1}(\tan x) = x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

Find the exact value of each expression, if it is defined.

Ex. 1: $\tan^{-1}(-1)$

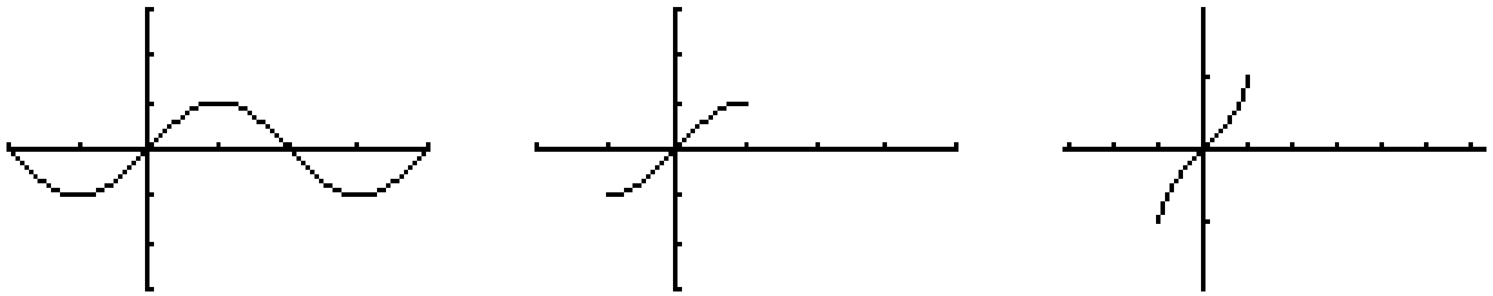
Ex. 2: $\tan^{-1}\left(\sqrt{3}\right)$

Ex. 3: $\tan\left(\tan^{-1} 42\right)$

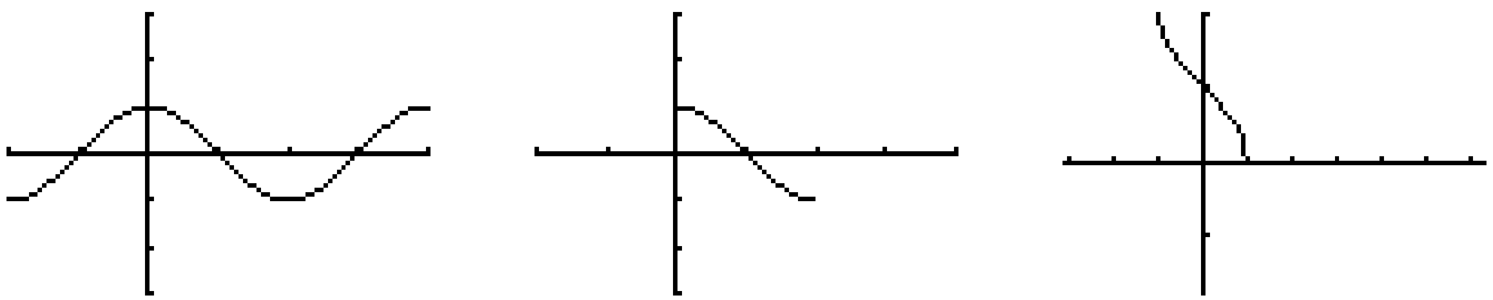
Ex. 4: $\tan^{-1}\left(\tan \frac{5\pi}{6}\right)$

Compare each trigonometric function with its inverse by looking at the graph of the function, the graph of the function when the domain is limited, and the graph of inverse trig function.

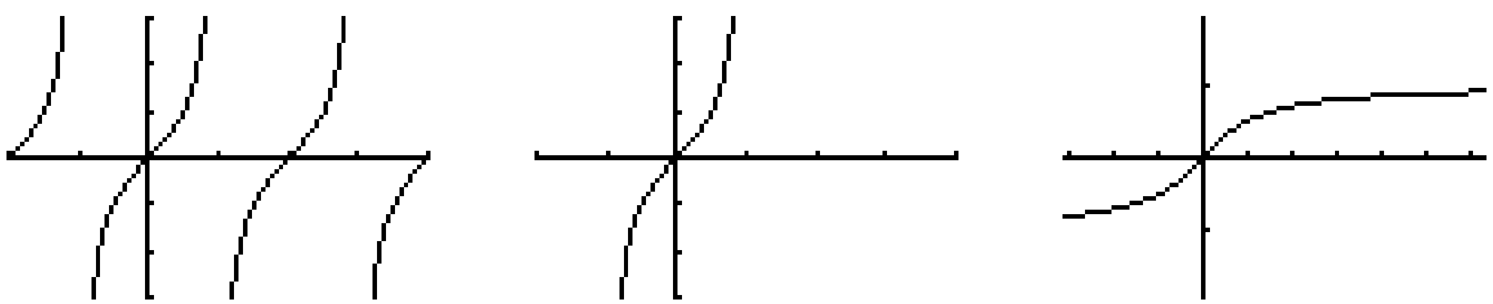
$y = \sin x$ $y = \sin^{-1} x$



$y = \cos x$ $y = \cos^{-1} x$



$y = \tan x$ $y = \tan^{-1} x$



Other inverse trigonometric functions

Inverse cotangent, inverse secant, and inverse cosecant functions exist. In particular, the limitations placed on secant and cosecant (in order to make them one-to-one) are awkward. Further, with the exception of inverse secant which shows up in some derivatives and integrals in calculus, they are not of significant use.

We will make limited use of these inverse functions, denoted as **cot**$^{-1}$ (or **arccot**), **sec**$^{-1}$ (or **arcsec**), and **csc**$^{-1}$(or **arccsc**), respectively.

Evaluate using a triangle.

Ex.: $\sin\left(\tan^{-1}\frac{3}{4}\right)$

Evaluate using an identity.

Ex.: $\cos\left(\sin^{-1}\frac{2}{3}\right)$

Evaluate.

Ex. 3: $\tan\left(\cos^{-1}\frac{5}{8}\right)$

Evaluate.

Ex. 4: $\sin\left(\tan^{-1}\left(\cos\left(\sin^{-1}\frac{12}{13}\right)\right)\right)$

Rewrite as an algebraic expression using a triangle.

Ex.: $\sin\left(\tan^{-1} x\right)$

Rewrite as an algebraic expression using an identity.

Ex.: $\sin\left(\cos^{-1} x\right)$

Rewrite as an algebraic expression.

Ex. 3: $\tan\left(\cos^{-1} x\right)$

Rewrite as an algebraic expression.

Ex. 4: $\sin\left(2\sin^{-1} x\right)$

Review of Complex Numbers

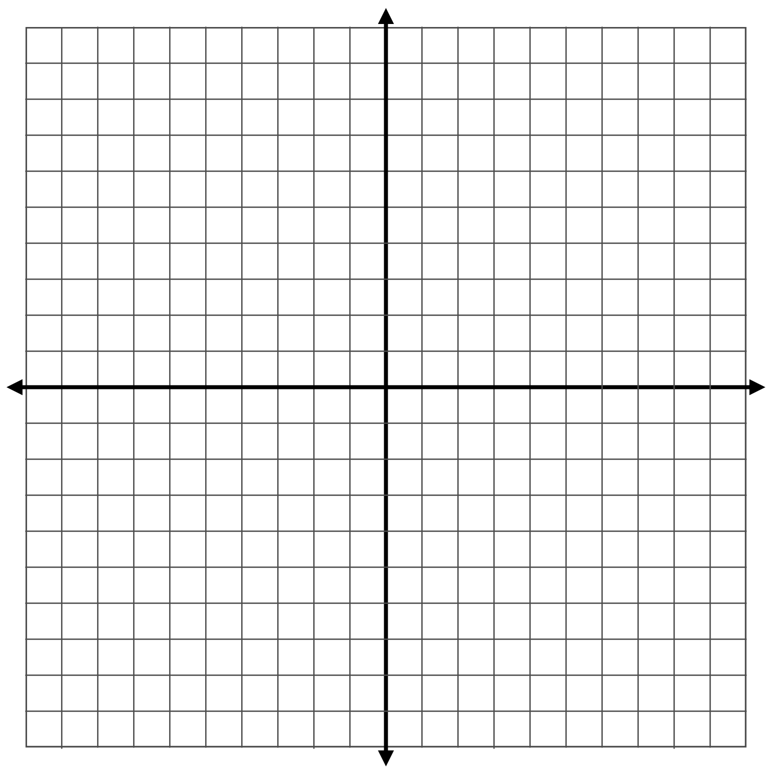
Definition: Let *z* be a complex number. Then $z = a + bi$, where *a* and *b* are real numbers and *i* is the imaginary root defined by $i = \sqrt{-1}$.

a is called the **real part** of the complex number while *bi* is called the **imaginary part** of the complex number.

Graph the real number *x* = 4 on an appropriate graph.

Graph the complex number *z* = 3 – 4*i* on an appropriate graph.

Representation of Complex Numbers on a Right Coordinate Plane



Absolute Value of a Complex Number

Definition: The absolute value of a complex number, also called the modulus of the complex number, is $|z| = \sqrt{a^2 + b^2}$

Calculate the modulus of *z*.

Ex.: $z = 3 - 4i$.

Note: The plural of modulus is moduli.

Polar Form of a Complex Number

Definition: The polar form of a complex number is defined as $z = r\left(\cos\theta + i\sin\theta\right)$, where $r = \left|z\right| = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$.

r is called the **modulus** of z while θ is called the **argument** of z.

In general, θ is not unique because there are infinitely many values of $\tan\theta$ which equal b/a, however we will restrict ourselves to values in $[0, 2\pi)$.

Find the polar form of z.

Ex. 1: $z = -2 + 2i$.

Find the polar form of z.

Ex. 2: $z = -\sqrt{3} - 3i$.

Find the polar form of z. As necessary, approximate five places after the decimal.

Ex. 3: $z = 3 - 4i$

Polar Coordinate Plane

Polar form of a complex number is associated with the concept of polar coordinates which identify the location of a point based on radius and angle.

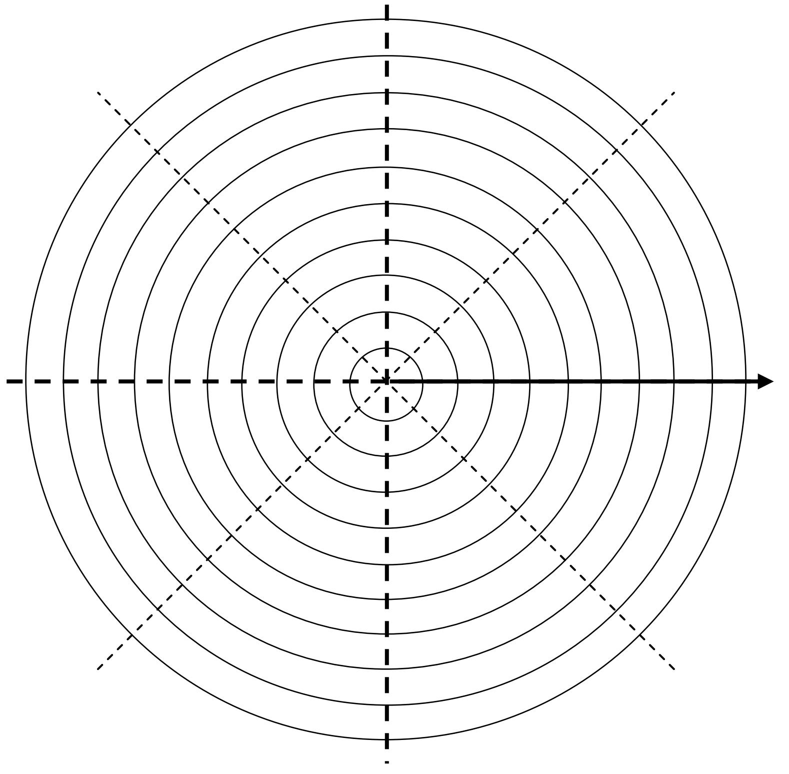
In polar form, coordinates are expressed (r, θ) where r is the radius from the origin and θ is an angle, usually expressed in radians, whose initial side is a horizontal line to the right of the origin.

We previously noted the connection between the complex number $a + b\boldsymbol{i}$ and the coordinates (a, b) on the right coordinate plane. In the polar form of a complex number, the form $r(\cos\theta + i\sin\theta)$ is associated with the coordinates (r, θ) on the polar coordinate plane.

Plot the polar form of a complex number on the polar coordinate plane below.

Ex. 1: $z = -2 + 2i$.

Ex. 2: $z = -\sqrt{3} - 3i$.



Multiplying and Dividing Complex Numbers;

Theorem: Given two complex numbers in polar form, $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, then their product and quotient can be found by:

$$z_1 \cdot z_2 = (r_1 \cdot r_2)\left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right)$$

and $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right)$.

Finding Powers and Roots of Complex Numbers

DeMoivre's Theorem

Theorem: If $z = r(\cos\theta + i\sin\theta)$, then for any integer *n*:

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

Theorem: If $z = r(\cos\theta + i\sin\theta)$ and *n* is any positive integer, then *z* has *n* distinct n^{th} roots generated by

$$r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right],$$

for all integers *k* from 0 to *n* – 1.

Find the product $z_1 \cdot z_2$ and quotient z_1/z_2 in polar form.

Ex.: $z_1 = -2 + 2i$ and $z_2 = -\sqrt{3} - 3i$

Find the square of z_1 and the cube root of z_2.

Ex.: $z_1 = -2 + 2i$ and $z_2 = -\sqrt{3} - 3i$

Find the polar form of z_1 and z_2 . Then find the cube of z_1 and the square root of z_2.

Ex. $z_1 = 1 - \sqrt{3}\, i$ and $z_2 = \sqrt{2} + \sqrt{2}\, i$.