Topic 6-1a: Exponential Functions

Definition: The exponential function with base \(a\) is defined by \(f(x) = a^x, \ a > 0\).

For \(a \neq 1\), the domain is \((-\infty, \infty)\) and the range is \((0, \infty)\).

Examples of Basic Exponential Functions

![Graphs of exponential functions](image)

Note that a power function is defined to be a function with a variable base \(x\) raised to a fixed exponent.

When a power function has an integer exponent, you get one of the previously covered basic functions (such as squaring and reciprocal).
Observations about exponential functions:

1. $f(0) = 1$ for all $a > 0$. (Basic exponential functions have $y$-intercepts of 1).

2. When $a > 1$, a basic exponential function is always increasing.
   When $0 < a < 1$, a basic exponential function is always decreasing.

3. At one end of the function displays asymptotic behavior dependent on the conditions in #2.

4. The closer $a$ is to 1, the more linear the graph appears.

End behavior of a basic exponential function

Recall that end behavior is the term used to describe what happens to function values as $x$ gets very large positive ($x \to \infty$) or very large negative ($x \to -\infty$). Further recall that an asymptote is a line that a graph approaches but never intersects.

Exponential functions and their end behavior can be divided into two groups based on the value of $a$:

When $a > 1$, the exponential function will increase without bound as $x \to \infty$ and will asymptotically approach 0 as $x \to -\infty$.

When $0 < a < 1$, the exponential function will asymptotically approach 0 as $x \to \infty$ and will increase without bound as $x \to -\infty$. 
The number $e$

Consider the special expression $\left(1 + \frac{1}{n}\right)^n$.

We know that, for any exponential expression with base greater than 1, the value of the expression will grow rapidly as $n$ gets very large positive.

We also know that 1 to any power is 1.

So what will happen to this expression as $n$ gets very large? Will the exponent being very large dominate and send the value toward infinity or will the base being very close to 1 dominate and send the value toward 1?

Definition: $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e \approx 2.71828182846$

$e$ is called Napier’s Constant, Euler’s Number, or the Natural Base.

Definition: The natural exponential function is an exponential function with base $e$:

$$f(x) = e^x.$$
Exponential Functions based on limited information

A limited amount of information is necessary to graphically define an exponential function. You need to know if the function has been dilated, what its \( y \)-intercept is, and one point other than the \( y \)-intercept.

Ex. 1 Find the exponential function of the form \( f(x) = a^x \) defined by the graph.

Ex. 2 Find the exponential functions of the form \( f(x) = C a^x \) defined by the graph.

Ex. 3 Find the exponential functions of the form \( f(x) = C a^x \) defined by the graph.
Topic 6-1b: Transformations of basic exponential functions

The algebraic transformations we covered in a previous unit are possible with exponential functions as well.

Ex. 1 Identify the basic function in the given function.

Determine the transformations applied to the basic function that produce the given function. State the transformations applied, in the correct order, using units and directions as appropriate.

Finally, determine the domain, range, and asymptote of the given function.

\[ f(x) = -3 + 2^{x-2} \]

Ex. 2 Identify the basic function in the given function.

Determine the transformations applied to the basic function that produce the given function. State the transformations applied, in the correct order, using units and directions as appropriate.

Finally, determine the domain, range, and asymptote of the given function.

\[ f(x) = 6 - 3^{x+1} \]
There are two primary formulas used for calculating the value of an account where interest is compounded:

**Formula 1 – For Discretely Compounded Interest**

\[
A(t) = P \left(1 + \frac{r}{n}\right)^{nt}
\]

**Formula 2 – Continuously Compounded Interest**

\[
A(t) = Pe^{rt}
\]

t = units of time in years
P = principle (initial value)
r = rate of interest as a decimal
A(t) = amount at/after time t
n = number of compounds per year

Ex. 1a Janice wishes to invest $5,000 for 10 years. Account A earns 4.95% interest compounded annually, account B earns 4.89% interest compounded monthly, and account C earns 4.87% interest compounded weekly. Determine how much would be made by investing in each account and identify the most profitable account.
Ex. 1b  At the last moment Janice discovers a fourth account, D, which earns 4.85% interest compounded continuously. How much would account D have after 10 years and would it be a better choice than accounts A, B, or C?
Ex. 2  A furniture company offers a promotion where no interest is accrued on an account for four years. In the fine print you find that if the account is not paid in full within the four year period, all the interest that could have accumulated on the original amount will retroactively be applied to the account. How much interest would be tacked on to the account if it is not paid off in four years if the purchase price of the furniture was $3000 and the interest rate is:

a: 15.99% interest compounded quarterly
b: 20.99% interest compounded daily
Topic 6-2b: Present Value of a Future Amount

Definition: The amount the principle must be to achieve a desired value at time $t$.

Ex: What is the present value of $10,000 for an account bearing 7.5% interest compounded monthly for 8 years?
Ex. 1

An abandoned island, overrun with rats, is being used as a dumping ground. The rat population is given by the function $n(t) = 45e^{0.15t}$, where $t$ is the number of years after 2005 and $n(t)$ is in thousands.

a: According to the formula, what was the population of rats on the island in 2005?

b: According to the formula, what is the growth rate in the rat population?

c: Estimate the number of rats on the island in the year 2020.
Ex. 2  A scientist wishes to do an experiment with fruit flies. At 9 am she has 200 fruit flies and under the controls of her experiment their population should increase at a rate of 35% per hour.

a: Write a function of the form
\[ n(t) = n_0 e^{rt} \]
to describe the growth of the population \( t \) hours after 9 am.

b: Estimate the population of fruit flies at 4 pm of the same day.

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Topic 6-3b: Application of Exponentials – Radioactive Decay

The amount of radioactive material in a sample can be determined by the formula \( m(t) = m_0 2^{\frac{t}{h}} \),

where:
- \( m_0 \) = initial amount of radioactive material
- \( m(t) \) = amount of radioactive material at time \( t \)
- \( t \) = units of time (defined by the half-life)
- \( h \) = half-life of radioactive material
Ex. 1  Modern smoke detectors use gamma rays produced by Americium-241 to help detect smoke at earlier stages. The half-life of Am-241 is 432 years. If 0.5 grams of Am-241 is used, how much will remain after 10 years when the life of the smoke detector is exhausted (due to mechanical and environmental wear)?

Ex. 2  A new treatment for cancer that targets bone metastasis involves the use of Radium-223. The half-life of Ra-223 is 11.43 days. What percentage of Ra-223 remains in the body after 28 days?
Topic 6-4a: Logarithms: The Inverse of an Exponential

So what is the inverse of an exponential function?

It should be easy to determine the inverse of each of these functions:

\[ f(x) = x + 3 \]
\[ g(x) = 7x \]
\[ h(x) = x^3 \]

Yet, we don’t have anything previously covered that serves as a proper inverse function to:

\[ f(x) = 2^x \]

Definition: Let \( a \) be a positive real number other than 1. Then the logarithm with base \( a \), denoted as \( \log_a \), is defined as follows:

\[ \log_a x = y \iff a^y = x \]

Observations: A logarithm with base \( a \) is the inverse operation of an exponential base \( a \).

The output of a logarithmic expression is an exponent.

Better said, a logarithmic expression \( \textit{is} \) an exponent.
Evaluate these basic logarithmic expressions by considering their analogues amongst exponential expressions:

<table>
<thead>
<tr>
<th>Ex. 1</th>
<th>( \log_7 1 )</th>
</tr>
</thead>
</table>

| Ex. 2 | \( \log_{11} 11 \) |

| Ex. 3 | \( \log_6 36 \) |

| Ex. 4 | \( \log_3 81 \) |

Properties of Logarithms

1. \( \log_a 1 = 0 \)

2. \( \log_a a = 1 \)

3. \( \log_a a^n = n \)

4. \( a^{\log_a n} = n \)
Evaluate these basic logarithmic expressions by applying the properties of logarithms and properties of exponents:

Ex. 1 \( \log_2 32 \)

Ex. 2 \( \log_{25} 5 \)

Ex. 3 \( \log_9 27 \)

Ex. 4 \( \log_8 4 \)

Ex. 5 \( \log_{12} \frac{1}{12} \)

Ex. 6 \( \log_4 \frac{1}{64} \)

Ex. 7 \( \log_{0.5} 16 \)

Ex. 8 \( \log_5 0.04 \)
Topic 6-4b: Logarithmic Functions & their Transformations

Basic logarithmic functions are logically the inverse functions to corresponding basic exponential functions.

Compare the graphs of the exponential function base two with the logarithmic function base two.

Definition: For a logarithmic function \( f(x) = \log_a x \), the domain is \((0, \infty)\), the range is \((-\infty, \infty)\), and a vertical asymptote exists at \(x = 0\).
Transformations are also possible with logarithmic functions.

Ex. 1  Identify the basic function in the given function.

Determine the transformations applied to the basic function that produce the given function. State the transformations applied, in the correct order, using units and directions as appropriate.

Finally, determine the domain, range, and asymptote of the given function.

\[ f(x) = -\log_4 (x + 2) \]

Ex. 2  Identify the basic function in the given function.

Determine the transformations applied to the basic function that produce the given function. State the transformations applied, in the correct order, using units and directions as appropriate.

Finally, determine the domain, range, and asymptote of the given function.

\[ f(x) = 5\log_3 (1 - x) \]
Logarithmic Functions based on limited information

Similar to exponential functions, it is possible to graphically define a logarithmic function using limited information. You need to know if the function has been dilated, what its $y$-intercept is, and one point other than the $y$-intercept.

**Ex. 1** Find the logarithmic function of the form $f(x) = \log_a x$ defined by the graph.

**Ex. 2** Find the logarithmic function of the form $f(x) = \log_a x$ defined by the graph.
Topic 6-4c: Special Logarithms

Common Logarithms: Base 10 logarithm

$$\log_{10} x \leftrightarrow \log x$$

$$y = \log x \leftrightarrow 10^y = x$$

Applications include pH scale, Richter scale, decibel scale, etc.

Natural Logarithms: Base e logarithm

$$\log_e x \leftrightarrow \ln x$$

$$y = \ln x \leftrightarrow e^y = x$$

Since the natural exponential function has so many uses, its inverse should logically have many uses as well.

Properties of Special Logarithms

<table>
<thead>
<tr>
<th></th>
<th>General</th>
<th>Common</th>
<th>Natural</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. $$\log_a 1 = 0$$</td>
<td>$$\log 1 = 0$$</td>
<td>$$\ln 1 = 0$$</td>
<td></td>
</tr>
<tr>
<td>2. $$\log_a a = 1$$</td>
<td>$$\log 10 = 1$$</td>
<td>$$\ln e = 1$$</td>
<td></td>
</tr>
<tr>
<td>3. $$\log_a a^n = n$$</td>
<td>$$\log 10^n = n$$</td>
<td>$$\ln e^n = n$$</td>
<td></td>
</tr>
<tr>
<td>4. $$a^{\log_a n} = n$$</td>
<td>$$10^{\log n} = n$$</td>
<td>$$e^{\ln n} = n$$</td>
<td></td>
</tr>
</tbody>
</table>
Topic 6-4d: More Log Concepts: Evaluating & Graphing

Using the properties of logarithms it should be relatively easy to evaluate $\log_2 32$.

Calculators will easily handle approximating $\log 32$ and $\ln 32$.

But how do we determine the approximate value of $\log_3 32$?

At the very least we should be able to determine between which two integers the value should fall.

Change of Base Formula

$$\log_a m = \frac{\log_b m}{\log_b a}$$

This formula allows us to calculate the approximate value of logarithms where the exact value is difficult or impossible to express as a non-logarithmic expression.

Since we can choose what we change the base into, the most logical choices are usually base 10 or base $e$, thus:

$$\log_a m = \frac{\ln m}{\ln a} = \frac{\log m}{\log a}$$

One very useful feature of the change of base formula is how it facilitates graphing.

In most calculators it’s impossible to directly enter $f(x) = \log_3 x$. But since $\log_3 x = \frac{\ln x}{\ln 3}$...
Topic 6-5: Laws of Logarithms

The laws of logarithms allow us to rewrite logarithmic expressions so they are easier to manipulate. Each law of logarithms has an analogue with the laws of exponents.

<table>
<thead>
<tr>
<th>Name</th>
<th>Law of Logarithms</th>
<th>Law of Exponents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Product-to-Sum</td>
<td>log_a(xy) = log_a x + log_a y</td>
<td>a^m \cdot a^n = a^{m+n}</td>
</tr>
<tr>
<td>Quotient-to-Difference</td>
<td>log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y</td>
<td>\frac{a^m}{a^n} = a^{m-n}</td>
</tr>
<tr>
<td>Power-to-Product</td>
<td>log_a(x^n) = n \log_a x</td>
<td>\left(a^m\right)^n = a^{m \cdot n}</td>
</tr>
</tbody>
</table>

Deconstructing a single logarithm using the laws of logarithms

By applying the laws and properties of logarithms, it is possible to rewrite a single complicated logarithm into a sum and/or difference of simpler logarithmic expressions and/or constants.

Ex. 1 Use the Laws of Logarithms so that no products, quotients, or powers (as possible) remain inside the logarithm. Simplify.

\[ \log_2(8x^4y) \]
Ex. 2  Use the Laws of Logarithms so that no products, quotients, or powers (as possible) remain inside the logarithm. Simplify.

\[ \ln \frac{a^2}{e c^5} \]

Ex. 3  Use the Laws of Logarithms so that no products, quotients, or powers (as possible) remain inside the logarithm. Simplify.

\[ \log_7 \sqrt[4]{v^2 w^3} \]
Ex. 4 Use the Laws of Logarithms so that no products, quotients, or powers (as possible) remain inside the logarithm. Simplify.

\[ \log_8 \frac{7\sqrt{x}}{y^6 z^2} \]

Ex. 5 Use the Laws of Logarithms so that no products, quotients, or powers (as possible) remain inside the logarithm. Simplify.

\[ \log \frac{10^x (x^2 - 1)}{x^3 (x^2 + 4)} \]
Reconstructing a single logarithm using the laws of logarithms

This is the opposite of the last skill. By applying the laws and properties of logarithms in reverse, it is possible to create take multiple logarithmic expressions and combine them to make a single logarithm.

Ex. 1 Rewrite as a single logarithm.
3log\(_{11}\) x + 4log\(_{11}\) y − 5log\(_{11}\) z

Ex. 2 Rewrite as a single logarithm.
log\(_{7}\) 2 − 6log\(_{7}\) u − log\(_{7}\) v

Ex. 3 Rewrite as a single logarithm.
ln3 + xln2 + 2lnx
Ex. 4 Rewrite as a single logarithm.  
\[ 3\log_2 x + \log_2 (x + 2) - \log_2 (x + 1) - \log_2 (x - 1) \]

Ex. 5 Rewrite as a single logarithm.  
\[ 2\log_4 x + \frac{1}{2}\log_4 (x + 1) - \log_4 3 - \log_4 (2x - 1) \]