DESCRIPTION OF RESEARCH INTERESTS

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1. Overview

My research focuses on the modular representation and cohomology theory of algebraic groups and related structures. The study requires algebraic tools from representation theory, homological algebra, commutative algebra, and computational algebra, as well as geometric methods from Lie theory and algebraic geometry. My research interests thus span the aforementioned areas.

Let G be a connected reductive algebraic group defined over $k = \overline{\mathbb{F}_p}$. For each positive integer r, the kernel of the r-th iterated Frobenius morphism $F_r : G \to G$ is called the r-th Frobenius kernel of G, denoted by $G_{(r)}$, and the subgroup of fixed points G^{F_r} is a finite group of Lie type, denoted by $G(\mathbb{F}_{p^r})$. Then for a rational G-module M, one can consider the restrictions of M to $G_{(r)}$ and to $G(\mathbb{F}_{p^r})$. There has been extensive study about connections and interactions between the three module categories of G, $G_{(r)}$, and $G(\mathbb{F}_{p^r})$, see for example [Hum]. It is observed that representations of the finite group of Lie type closely resemble those of Frobenius kernels, however, one could not establish a direct link between the two categories. Exploring relationships between these categories is the current theme of my work.

Very little is known about the cohomology of Frobenius kernels $G_{(r)}$. Most of the work has been done for $r = 1$ due to the connection between the first Frobenius kernels and the restricted Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. In particular, Friedlander and Parshall proved that the cohomology ring of $G_{(1)}$ is isomorphic as an algebra to the coordinate ring of the nilpotent cone of $\mathfrak{g}$ when the prime p is large enough [FP]. In joint work with Drupieski and Nakano [DNN], we completely described the cohomology ring structure for the first Frobenius kernel of maximal unipotent subgroups U of a simple algebraic group. This computation provides an answer to a 25-year-old question concerning the ring structure of $\mathrm{H}^\bullet(U_1, k)$. For higher values of r, no concrete results have been known except my computations for SL_2 case in [Ngo1]. In a recent work with Lux and Zhang, we applied these results to calculate the cohomology of Weyl modules over SL_2 [LNZ]. Remarkable papers of Suslin, Friedlander, and Bendel [SFB1][SFB2] reduces the study of the cohomology for $G_{(r)}$ to that of *restricted nilpotent commuting varieties* of r-tuples. In collaboration with Levy and Šivic, we have explored many interesting properties of those objects, hence provided some insights to the cohomology of Frobenius kernels, see [NS][LN][LNS]. Our recent progress shows a direct connection between the cohomology of $G_{(r)}$ and $G(\mathbb{F}_{p^r})$.

Commuting varieties of r-tuples over matrices are of central interests in algebraic geometry and commutative algebra. Especially in the case $r = 2$, the long-standing conjecture stating that the defining ideal of such a variety radical and its coordinate ring is Cohen-Macaulay has been opened for decades. My recent work on commuting varieties makes some contributions to this problem, see [Ngo2][Ngo3].

2. Cohomology of SL_2

Let $G = SL_2$ be the rank one simple algebraic group defined over an algebraically closed field k of characteristic $p > 0$. Fix a maximal torus T of G, we write $X(T)$ for the weight lattice of T, and let ϖ be the fundamental weight in $X(T)$. It then follows that $X(T) = \mathbb{Z}\varpi$. Abbreviating $n\varpi$ by n for each $n \in \mathbb{Z}$, we identify $X(T)$ with $\mathbb{Z}$. It follows that X^+, the set of dominant weights in $X(T)$, is $\mathbb{N}$ under this identification. For each $m \in X^+$, the induced module of highest weight m is denoted by $\mathrm{H}^0(m)$, hence the corresponding Weyl module $V(m)$ is defined to be the dual of $\mathrm{H}^0(m)$. Every simple G-module $L(m)$ is isomorphic to the socle of an induced module $\mathrm{H}^0(m)$.

2.1. **Summary of past work.** In [Ngo1], I computed the cohomology of $G_{(r)}$ with coefficients in $\mathrm{H}^0(m)$ for all $r \geq 1, m \geq 0$. The results are encoded in the number of integral solutions of certain linear equations. Then I proved in [Ngo4] that the cohomology ring of $G_{(r)}$ is Cohen-Macaulay, hence satisfies the Poincare's duality. I showed further that this property does not hold for higher rank groups.

2.2. **Recent results.** Using my calculations in [Ngo1] and the connection between the cohomology of Frobenius kernels and that of algebraic groups established by work of Cline, Parshall, and Scott [CPS], Lux, Zhang, and I compute exactly the dimension of cohomology space of G with coefficients in $V(m)$. Explicitly, given $m \in \mathbb{N} = X^+$, let r_m be the largest positive integer such that $p^{r_m} \leq m$ and let

$$c_0 + c_1 p + \cdots + c_{r_m} p^{r_m}$$

be the p-adic expansion of m with $c_i \in \{0, \ldots, p-1\}$. The height of m, denoted by $\mathrm{ht}(m)$, is defined by $\sum_{i=0}^{r_m} c_i$. We denote further by s_m the number of zeros in the sequence $\{c_1, \ldots, c_{r_m}\}$. Hence, $r_m - s_m$ is the number of non-zero c_i's with $1 \leq i \leq r_m$. We then have

Theorem 2.1. [LNZ, Theorem 4.2 and Proposition 4.8] *For all $m, n \geq 0$, if $n \leq 2p-3$ then we have*

$$\dim \mathrm{H}^n(G, V(m)) = \binom{r_{\frac{m}{2}+1} - s_{\frac{m}{2}+1}}{2\mathrm{ht}(\frac{m}{2}+1) - c_0 - n - 1}.$$

Consequently, for all $n \leq 2p-3$, we have

$$\dim \mathrm{H}^n(G, V(m)) \leq F(n+1)$$

where $F(n+1)$ is the $(n+1)$-th Fibonacci number.

We are then able to calculate the exact dimension of the third degree extension spaces between any two Weyl modules [LNZ, Theorem 5.1] using the above formula together with recursive formula for extension spaces in [Pa]. Consequently, we found that these dimensions are bounded from above by 3 [LNZ, Corollary 5.2]. This result extends the work of Cox and Erdmann in [CE]. Employing the theory of partition functions in Number theory, we further obtain upper bounds for cohomology dimension of various groups of rank 1. In particular, let $C_n = n4^n e^{2\pi n/\sqrt{3}}$, we have

Theorem 2.2. [LNZ, Corollary 4.3, Theorems 6.5 and 6.8] *For all $m \geq 0, n \geq 1$,*

$$\dim \mathrm{H}^n(G, V(m)) \leq C_{n+1}$$

and

$$\dim \mathrm{H}^n(G, L(m)) \leq (2n+7)C_n.$$

Let $G(\mathbb{F}_{p^r})$ the finite group of Lie type associated to G and L be any simple module of $G(\mathbb{F}_{p^r})$. Then

$$\dim \mathrm{H}^n(G(\mathbb{F}_{p^r}, L) \leq (2n+7)C_n.$$

The proof is rather interesting as we established the inequality for $G(\mathbb{F}_{p^r})$ first and then using generic cohomology and recursive formulas in [Pa] to obtain the upper bound for $\dim \mathrm{H}^n(G, L(m))$. The theorem also implies that the sequence

$$\Big(\max\{\dim \mathrm{H}^n(SL_2, M) : M = V(m) \text{ or } L(m)\}\Big)_{n=1}^{\infty}$$

grows exponentially. This strengthens the main result in [EHP, Corollary 5.6.3] and in [Ste].

2.3. **Future investigations.** I am working on generalizing Theorem 2.2 to arbitrary algebraic group. In particular, I would like to find a universal bound for G, which only depends on the degree n and the dimension of the coefficient module M. The starting point is to find a linear bound for the B-cohomology as questioned in [BNPPSS]. I believe this could be done by calculation methods in [LNZ].

3. Commuting varieties of r-tuples

Assume $\mathfrak{g}$ is a simple Lie algebra over the field k (defined in the previous section). Let $\mathcal{N}_1(\mathfrak{g})$ and $\mathcal{N}(\mathfrak{g})$ be the restricted nullcone and nilpotent cone of $\mathfrak{g}$. Note that $\mathcal{N}_1(\mathfrak{g}) = \mathcal{N}(\mathfrak{g})$ for $p \geq h$, the Coxeter number of $\mathfrak{g}$. For each $r \geq 1$, define for each closed subvariety V of $\mathfrak{g}$

$$C_r(V) = \{(x_1, \ldots, x_r) \in V^r : [x_i, x_j] = 0\ ,\ 1 \leq i, j \leq r\}$$

the commuting variety of r-tuples in V. When $V = \mathcal{N}_1(\mathfrak{g})$ or $\mathcal{N}(\mathfrak{g})$, we call $C_r(V)$ a (restricted) nilpotent commuting variety.

3.1. **Summary of past work.** The study of nilpotent commuting varieties started not long ago. The pioneering work of Premet [Pr] showed that $C_2(\mathcal{N}(\mathfrak{g}))$ has pure dimension[1] $\dim \mathfrak{g}$. In [Ngo4], I could show that the result does not hold for arbitrary r. In joint work with Šivic [NS], we determined the (ir)reducibility of the variety $C_r(\mathcal{N}(\mathfrak{sl}_n))$ for various values of n and r. Explicitly, it is reducible for all $n, r \geq 4$. Moreover, for $r = 3$, it is irreducible for all $n \leq 6$. For $n = 2$, I proved further that the varieties $C_r(\mathfrak{gl}_2)$ and $C_r(\mathcal{N}(\mathfrak{sl}_2))$ have nice geometric properties such as Cohen-Macaulayness and having rational singularities for all $r \geq 1$ and $p \neq 2$ [Ngo2].

3.2. **Work in progress.** In a joint work with Levy, we investigate mixed commuting varieties

$$C(\overline{\mathcal{O}}, \mathcal{N}(\mathfrak{g})) = C_2(\mathcal{N}(\mathfrak{g})) \cap \overline{\mathcal{O}} \times \mathcal{N}(\mathfrak{g})$$

for each orbit $\mathcal{O}$ in $\mathcal{N}(\mathfrak{g})$. These objects can be identified with the support varieties of modules over the second Frobenius kernel of G, see [SFB2]. For exceptional types (except E_8) and sufficiently large p, we explicitly described $C(\overline{\mathcal{O}}, \mathcal{N}(\mathfrak{g}))$ for each orbit $\mathcal{O}$ in $\mathcal{N}$ by computer calculations (GAP). The problem turns out to be more challenging for classical types of $\mathfrak{g}$.

[1]This means all irreducible components are of the same dimension.

Suppose e is a nilpotent element in $\mathfrak{g}$ and $\mathcal{O}_e$ corresponds to a partition $[\alpha_1, \dots, \alpha_s]$. Then define

$$r_e = \begin{cases} s-1 & \text{if } G \text{ is of type } A, \\ \frac{s-t}{2} & \text{if } G \text{ is of type } C, \\ \frac{s-t'}{2} & \text{if } G \text{ is of type } B \text{ or } D, \end{cases}$$

where t (resp. t') is the number of distinct even (resp. odd) α_i in the partition with odd multiplicity. Using this formula and the idea of Premet, we compute the dimension of mixed commuting varieties as follows.

Theorem 3.1. [LN] *Suppose $\mathfrak{g}$ is a classical simple Lie algebra and p is a good prime. For each nilpotent element e in $\mathfrak{g}$, one has*

$$\dim C(\overline{\mathcal{O}_e}, \mathcal{N}(\mathfrak{g})) = \dim \mathfrak{g} - \min_{\mathcal{O}_{e'} \le \mathcal{O}_e} \{r_{e'}\}.$$

In particular, $\dim C(\overline{\mathcal{O}_e}, \mathcal{N}(\mathfrak{sl}_n)) = \dim \mathfrak{g} - r_e$ *and* $C(\overline{\mathcal{O}_{\mathrm{sub}}}, \mathcal{N}(\mathfrak{sl}_n))$ *is equidimensional.*

If e is a square zero element, i.e. $\mathcal{O}_e$ corresponds to a partition containing only 2's and 1's, then we can describe exactly such varieties.

Theorem 3.2. [LN] *Retain the assumption of $\mathfrak{g}$ as above. For each square zero element e, one has*

$$C(\overline{\mathcal{O}_e} \times \mathcal{N}) = \bigcup_{\substack{\mathcal{O}_{e'} \le \mathcal{O}_e \\ r_{e'} = r_e}} C(e'),$$

where $C(e') = \overline{G \cdot (e', z(e') \cap \mathcal{N}(\mathfrak{g}))}$, *with $z(e')$ the centralizer of e' in $\mathfrak{g}$.*

Let G be a simple, simply-connected algebraic group such that $\mathrm{Lie}(G) = \mathfrak{g}$. A closely related object to commuting varieties of r-tuples is the variety $G \cdot V$ where V is a B-stable closed subvariety in a B-module M. In an on-going project, I could prove the following

Theorem 3.3. [Ngo3] *Suppose that the characteristic of k is 0. Let $I(V) = \langle f_1, \dots, f_s \rangle$ be the defining ideal of V in $k[M] = S(M^*)$, the symmetric algebra over M^*, with f_i of weight β_i for $1 \le i \le s$. Assume that*

$$m : G \times^B M \to G \cdot M$$

is a rational resolution of $G \cdot M$. If V is normal and for all $i, r \ge 1$

$$R^i \operatorname{ind}_B^G (S(M^{*r}) \otimes \beta_j) = 0$$

for all $1 \le j \le s$, then $G \cdot V$ is normal. If, in addition, the variety V has rational singularities, then so does $G \cdot V$.

This theorem generalizes the main result of Kempf in [K]. It then follows that the commuting varieties $C_r(\mathcal{N}(\mathfrak{sl}_3))$ and $C_2(\mathfrak{gl}_3)$ have rational singularities, hence Cohen-Macaulay, see [Ngo3]. The theorem also provides a new approach in proving that the commuting variety $C_2(\mathfrak{g})$ has rational singularities.

3.3. **Future investigations.** Here are a couple of my current research projects in this area:

(1) Levy and I have been trying to show that $C(\overline{\mathcal{O}}, \mathcal{N}(\mathfrak{sl}_n))$ is equidimensional for each $n \geq 1$ and orbit $\mathcal{O}$ in $\mathfrak{sl}_n$. Note that this fact does not hold for other classical types as we found counter-examples already. In a long shot, we hope to describe $C(\overline{\mathcal{O}}, \mathcal{N}(\mathfrak{g}))$ for arbitrary orbit $\mathcal{O}$ and $\mathfrak{g}$.

(2) My work with Šivic in [NS] could be extended further. Indeed, we are interested in determining exactly the values of n for which $C_3(\mathcal{N}(\mathfrak{sl}_n))$ is (ir)reducible.

4. Connection to finite simple groups of Lie type

4.1. This is a novel aspect of my joint work with Levy and Šivic. In particular, while studying the cohomology ring of Frobenius kernels via properties of nilpotent commuting varieties $C_r(\mathcal{N}(\mathfrak{g}))$, we have found a surprising result:

Theorem 4.1. [LNS] *Suppose $G = GL_n$. For all $r \geq 8, n \geq 4$, and $p > 3$, we have*

$$\dim \mathrm{H}^{ev}(G_{(r)}, k) = \left(\frac{r+1}{r}\right) \dim \mathrm{H}^{ev}(G(\mathbb{F}_{p^r}), k).$$

Consequently, if M is a rational G-module whose dimension is not divisible by p, then

$$c_{G_{(r)}}(M) = \left(\frac{r+1}{r}\right) c_{G(\mathbb{F}_{p^r})}(M),$$

where $c_H(M)$ denotes the complexity of the module M over a group H

As the complexity indicates the rate of growth of the sequence of cohomology dimensions, the result shows the similarity of cohomological behaviors of Frobenius kernels and finite simple groups of Lie type. Unfortunately, the identities do not hold for any type of G as we found counter-examples when G is of type B or G_2. However, we could prove the following

Theorem 4.2. [LNS] *Let G be a simple algebraic group defined over k. Suppose that the characteristic p is good for G. Then for any G-module M, we have*

$$c_{G_{(r)}}(M) \geq \left(\frac{r+1}{r}\right) c_{G(\mathbb{F}_{p^r})}(M).$$

The inequality significantly generalizes the main result of Lin and Nakano in [LNa] on the relation between the first Frobenius kernel $G_{(1)}$ and the finite simple group of Lie type $G(\mathbb{F}_p)$. It follows that if M is $G_{(r)}$-projective then M is projective over $G(\mathbb{F}_{p^r})$, hence we obtain an alternative proof for the main result in Drupieski's paper [D]. The proof for the theorem mainly relies on the aforementioned result of Lin and Nakano, and the geometric property below.

Theorem 4.3. [LNS] *Let W be a B-stable closed subvariety of $\mathfrak{u}^r$ for some positive integer r. Then we have* $\dim G \cdot W \geq \dfrac{r+1}{r} \dim W$.

4.2. **Future investigations.** We are working to extend Theorem 4.1 to the case when G is of type C or D. In other words, we aim to prove the following

Conjecture 4.4. *Let G be a simple classical algebraic group of type A, C, or D. Suppose p is good for G. Then for* $\operatorname{rank}(G)$ *and r sufficiently large, we have*

$$c_{G_{(r)}}(k) = \left(\frac{r+1}{r}\right) c_{G(\mathbb{F}_{p^r})}(k).$$

Note that the conjecture holds for type A for $p \neq 3$ from Theorem 4.1. It is also interesting to determine when it holds for other types.

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