NEW METHOD TO DESCRIBE THE 2-DIMENSIONAL COHOMOLOGY OF THE PRODUCT OF TWO GROUPS

NGO VO NHAM

(The university of Natural science at Ho Chi Minh city)

In this paper, our main purpose is to show a new method applying to caculate the 2 – dimensional cohomology group of the product of two groups.

Let Π, Π' be two multiplicative groups, A is a (Π×Π')- module. Then A is also a Π-module (or Π'-module) induced by the injection $i:\Pi \to \Pi\times\Pi'$ (or $i':\Pi' \to \Pi\times\Pi'$). We define h, a map from Π×Π' to A such that

$$h(1,y)=h(x,1)=0 \quad \text{with} \quad \forall x\in\Pi, y\in\Pi'.$$

Theorem A. *There exists a isomorphism*

$$\theta : H^2(\Pi\times\Pi',A)\cong H/\delta H \quad ,$$

where

$H=\{(f_2,f_2',h)\}$, $f_2\in Z^2(\Pi,A), f_2'\in Z^2(\Pi',A)$ such that $\forall$x,x'$\in\Pi$, y,y'$\in\Pi'$,

$$h(xx',y)+f_2(x,x')=(1,y)f_2(x,x')+h(x,y)+(x,1)h(x',y), \tag{1}$$

$$h(x,yy')-f_2'(y,y')=-(x,1)f_2'(y,y')+h(x,y)+(1,y)h(x,y'). \tag{2}$$

And

$\delta H=\{(\delta g_2,\delta g_2',\delta h(g_2,g_2'))\}$, $\quad \delta g_2\in B^2(\Pi,A), \delta g_2'\in B^2(\Pi',A)$ with

$g_2:\Pi\to$ A, $\; g_2':\Pi'\to$ A such that $g_2(1)=g_2'(1)=0$, and

$$\delta h(g_2,g_2')(x,y)=(x,1)g_2'(y)+g_2(x)-(1,y)g_2(x)-g_2'(y), \quad \forall(x,y)\in\Pi\times\Pi'. \tag{3}$$

Proof. Since a 2-dimensional cochain $F\in Z^2(\Pi\times\Pi',\mathrm{A})$ must does satisfy the identity

$$XF(Y,Z)+F(X,YZ)=F(X,Y)+F(XY,Z), \qquad X,Y,Z\in\Pi\times\Pi',$$

we can expand

$$F((x,y),(x',y'))=F((x,1),(x',1))+(xx',1)F((1,y),(1,y'))-[F((x,1),(1,y))$$
$$-F((xx',1),(1,yy'))+(x,y)F((x',1),(1,y'))]-(x,1)[F((x',1),(1,y))-F((1,y),(x',1))].$$

Let

$$f_2(x,x')=F((x,1),(x',1)),\;\; f_2(y,y')=F((1,y),(1,y')),$$
$$h(x',y)=F((x',1),(1,y))-F((1,y),(x',1)) \text{ and}$$
$$G(x,y)=F((x,1),(1,y))$$

$$\Rightarrow F((x,1),(1,y))-F((xx',1),(1,yy'))+(x,y)F((x',1),(1,y'))=G(x,y)-G(xx',yy')$$
$$+(x,y)G(x',y')$$
$$=\delta G((x,y),(x',y')).$$

Hence,

$$F((x,y),(x',y'))=f_2(x,x')+(xx',1)f_2'(y,y')-(x,1)h(x',y)-\delta G((x,y),(x',y'),$$
$$\forall(x,y),(x',y')\in\Pi\times\Pi'.$$

We can easily prove that $f_2\in Z^2(\Pi,\mathrm{A}), f_2'\in Z^2(\Pi',\mathrm{A})$ and h satisfy identities (1), (2). Therefore, we define a map θ such that $\theta(F) = (f_2, f_2', h)$. We can verify that θ is a isomorphism.□

This theorem is very important in our paper. Although it seems very complex, it is also convenient in some cases. Applying it, we describe the 2-dimensional cohomology of the product of two free groups. First, we recall the Lemma in [1].

Lemma 7.2. (Maclane – Homology, chapter IV)

If F is a free group with generators e_i, $i \in J$, A is a F-module then the group of all maps $h: F \to A$ such that always

$$h(xy) = xh(y) + h(x), \qquad \forall x, y \in F$$

is isomorphic to the product $\prod_{i \in J} A_i$ under the correspondence which sends each h to the family $\{h(e_i)\}_{i \in J}$ in $\prod_{i \in J} A_i$.

Using the same method to prove, we can generalize this result.

Theorem B. *If F_1, F_2 are free groups with generators $\{e_i\}_{i \in I}, \{w_j\}_{j \in J}$, A is a ($F_1 \times F_2$)–module then the group of all maps $h: F_1 \times F_2 \to A$ such that always*

$$h(xx', y) = h(x, y) + (x,1)h(x', y), \qquad \forall x, x' \in F_1, y \in F_2,$$
$$h(x, yy') = h(x, y) + (1, y)h(x, y'), \qquad \forall x \in F_1, y, y' \in F_2,$$

is isomorphic to the product $\prod_{I \times J} A$, under the correspondence which sends each h to the family $\{h(e_i, w_j)\}_{I \times J}$.

By use of these Theorem, we obtain the following

$$H^2(F_1 \times F_2, A) \cong \prod_{I \times J} A \Big/ \prod_{I \times J} S_{ij},$$

where $S_{ij} = \{a_j - e_i a_j - a_i + w_j a_i \mid a_i, a_j \in A\}$ *with* $i \in I, j \in J$.

Proof. Since $H^2(F_1, A) = H^2(F_2, A) = 0$, from Theorem A we have

$$H(F_1 \times F_2, A) \cong K \big/ \delta K,$$

where K is a group of all maps $h: F_1 \times F_2 \to A$ such that

$$h(xx', y) = h(x, y) + (x,1)h(x', y), \qquad \forall x, x' \in F_1, y \in F_2,$$
$$h(x, yy') = h(x, y) + (1, y)h(x, y'), \qquad \forall x \in F_1, y, y' \in F_2,$$

and δK is a subgroup of K including all maps $\delta h(g_2, g_2')$ as (3). Using Theorem B and the Lemma 7.2 in [1], we can describe that $K \cong \prod_{I\times J} A$ and $\delta K \cong \prod_{I\times J} S_{ij}$.□

When A is a trivial module, we have an interesting result. First, we define $D(\Pi, \Pi', A)$ is a set of all maps $h : \Pi \times \Pi' \to A$ such that

$$h(xx', y) = h(x, y) + h(x', y), \qquad \forall x, x' \in \Pi, y \in \Pi',$$

$$h(x, yy') = h(x, y) + h(x, y'), \qquad \forall x \in \Pi, y, y' \in \Pi'.$$

From Theorem A, the following result is proved by induction

$$H^2(\prod_{i=1}^{n} \Pi_i, A) \cong \prod_{i=1}^{n} H^2(\Pi_i, A) \times \prod_{i<j}^{n} D(\Pi_i, \Pi_j, A),$$

Using this formula, we can easily calculate

$$H^2(\prod_{i=1}^{n} C_{m_i}, A) \cong \prod_{i=1}^{n} A/m_i A \times \prod_{i<j}^{n} A_{(m_i, m_j)},$$

where C_i is the cyclic group of order m_i, (m_i, m_j) is the greatest common divisor of m_i, m_j, and $A_{(m_i, m_j)} = \{a \in A \mid (m_i, m_j)a = 0\}$.

References

1. **Sauders MacLane**, *Homology,* Springer Verlag (1963).
2. **Joseph J. Rotman**, *An introduction to the Theory of groups,* Springer Verlag (1995).
3. **Leonard Evens**, *The cohomology of groups,* Clarendon press (1991).