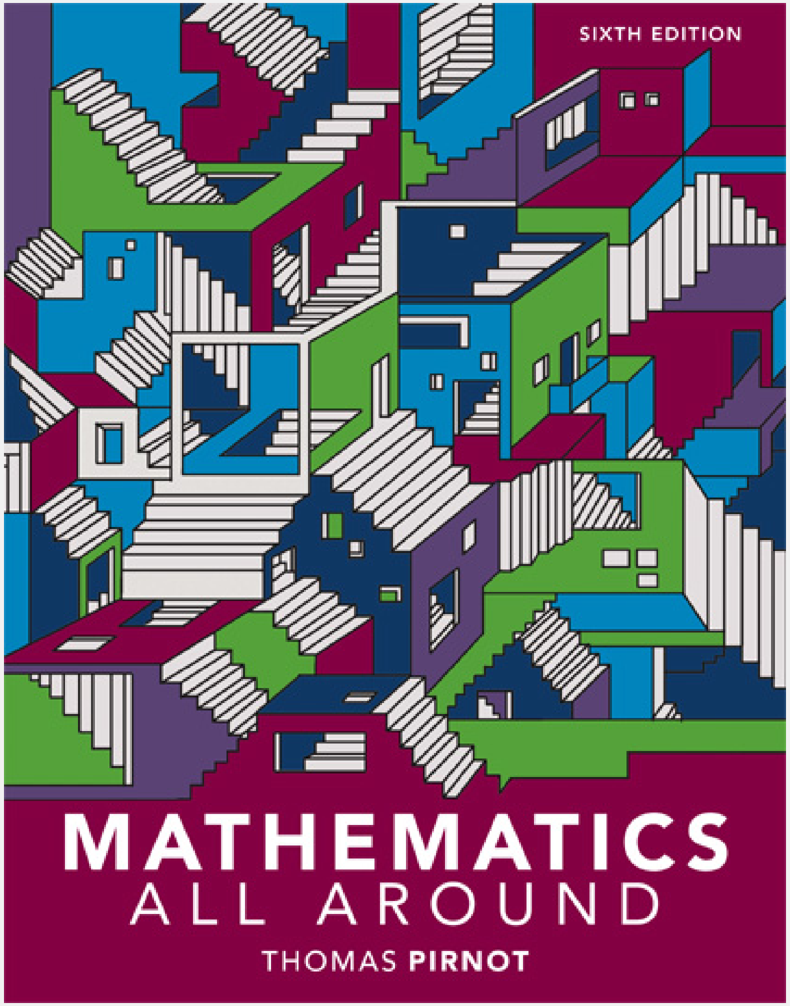
7.4

Exponential Models



7.4 Exponential Equations and Growth

- Explain the difference between linear, quadratic, and exponential growth
- Use exponential equations to model growth.
- Solve exponential equations using the log function.
- Use logistic models to describe growth.

Exponential Growth

Suppose you invest $1000 (the *principal*) into an account paying 8% interest per year.

After the first year:

$$\$1{,}000 + \underbrace{8\% \times \$1{,}000}_{\text{Interest}} = \$1{,}000 + \underbrace{0.08 \times \$1{,}000}_{\text{Interest}}$$

$$= \$1{,}000 + \underbrace{\$80}_{\text{Interest}} = \$1{,}080.$$

amount at end of first year

Exponential Growth

After the second year:

$$\$1{,}080 + \underbrace{8\% \times \$1{,}080}_{\text{Interest}} = \$1{,}080 + \underbrace{0.08 \times \$1{,}080}_{\text{Interest}}$$

$$= \$1{,}080 + \underbrace{\$86.40}_{\text{Interest}} = \$1{,}166.40.$$

amount at end of second year

Exponential Growth

To compute the amount for successive years, we perform similar calculations.

The process of earning interest on interest is called *compounding*.

Exponential Growth

Year	Beginning Balance	+	Interest for Current Year	=	Balance at End of Year
1	1,000	+	$0.08 \times 1{,}000$	=	$1{,}000(1.08) = 1{,}080$
2	1,080	+	$0.08 \times 1{,}080$	=	$1{,}080(1.08) = 1{,}000(1.08)(1.08)$
3	1,166.40	+	$0.08 \times 1{,}166.40$	=	$1{,}166.40(1.08) = 1{,}000(1.08)(1.08)(1.08)$

Exponential Growth

The Compound Interest Formula

If we invest the amount *P*, called the *principal*, in an account earning a yearly interest rate *r* and we compound the interest for *n* years, then the amount in the account, *A*, is

$A = P(1 + r)^n$.

Example: Using the Compound Interest Formula

Suppose that you deposit \$1,000 in an account that is compounded annually at a rate of 8%. Find the amount in this account after 30 years.

Solution

P = \$1,000, $n = 30$, and $r = 0.08$

$A = P(1 + r)^n$

$= 1{,}000(1 + 0.08)^{30}$

$= 1{,}000(1.08)^{30} = \$10{,}062.66.$

Exponential Growth

An **exponential equation** is an equation of the form

$$y = a \cdot b^x.$$

The compound interest formula is one example of an exponential equation.

Example: Modeling Population Growth with an Exponential Function

According to the U.S. Census Bureau, in 2014 the U.S. population was approximately 319 million, with an annual growth rate of 0.6%. If this growth rate continues until 2053, what will the population of the United States be then?

Solution

P = 319 million, r = 0.006,
n = 2053 – 2014 = 39

$$P(1+r)^n = 319(1+0.006)^{39}$$

$$= 319(1.006)^{39} = 402.8 \text{ million}$$

Exponential Models

We use the log function to find the time it takes a quantity that is growing exponentially to double.

The function log(*a*) returns the answer to the question, "What power *n* must 10 be raised to in order to get a value of *a*?"

$$a = 10^n$$

Exponential Models

Exponent property of the log function

$$\log y^x = x \log y$$

Example: Using the Log Function to Solve an Equation

Solve the equation $5 = 3^x$.

Solution

Take the log of both sides. $\log 5 = \log 3^x$

Use the exponent property. $\log 5 = x \log 3$

Divide both sides by log 3. $\dfrac{\log 5}{\log 3} = x$

$$\frac{0.69897}{0.47712} \approx 1.46$$

Example: Doubling a Population

In 2016, India's population was 1.3 billion, with an annual growth rate of 1.2% (*Source:* worldometers.info). Assuming that the growth rate will remain the same, in what year will India's population double?

Solution

We will use the growth model $A = P(1 + r)^n$, where $P = 1.3$, $r = 0.012$, and A is the future population of 2.6.

$$2.6 = 1.3(1+0.012)^n$$

Example: Doubling a Population (cont)

Solve.

$$2.6 = 1.3(1 + 0.012)^n$$

$$2 = (1.012)^n$$

$$\log 2 = \log(1.012)^n$$

$$\log 2 = n \log(1.012)$$

$$n = \frac{\log 2}{\log 1.012} \approx 58.1$$

If the growth rate were to continue at 1.2%, the population would double in 58.1 years, or by 2075.