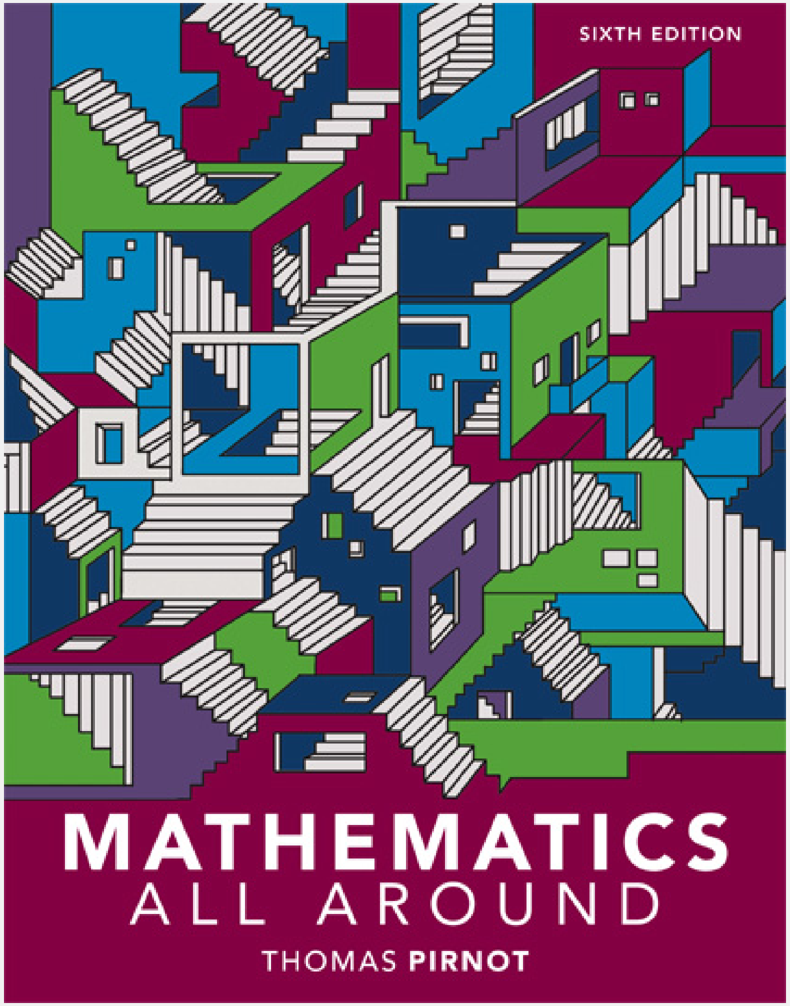
3.3

Conditional and Biconditional Logic



3.3 The Conditional and Biconditional

- Construct truth tables for conditional statements.
- Identify logically equivalent forms of a conditional.
- Use alternative wording to write conditionals.
- Construct truth tables for biconditional statements.

Conditional

In order to understand when a conditional statement is true or false, consider this example.

Mr. Gates, the owner of a small factory, has a rush order that must be filled by next Monday and he approaches you with this generous offer:

If you work for me on Saturday, then I'll give you a $100 bonus.

If we let w represent "You work for me on Saturday" and b represent "I'll give you a $100 bonus," then this statement has the form $w \rightarrow b$.

Conditional (cont)

We must examine four cases to determine exactly when Mr. Gates is telling the truth and when he is not.

Case 1 (*w* is true and *b* is true.):

You come to work and you receive the bonus. In this case, Mr. Gates certainly made a truthful statement.

Case 2 (*w* is true and *b* is false.):

You come to work and you don't receive the bonus.

Conditional (cont)

Mr. Gates has gone back on his promise, so he has made a false statement.

Case 3 (*w* is false and *b* is true.):

You don’t come to work, but Mr. Gates gives you the bonus anyway.

Conditional (cont)

Think carefully about this case because it differs from the way we tend to use *if* . . . *then* in everyday language. When you listen to Mr. Gates, do not read more into his statement than he actually said. You do not expect to get the bonus if you did not come to work because that is your experience in everyday life. However, Mr. Gates never said that. *You are assuming this condition.* Remember that in logic, a statement is either true or false. Therefore, because Mr. Gates did not say something false, he has told the truth.

Conditional (cont)

Case 4 (*w* is false and *b* is false.):

You don't come to work and you don't receive the bonus.

In this case, Mr. Gates is telling the truth for exactly the same reason as in Case 3.

Because you did not come to work, Mr. Gates can give you the bonus or not give you the bonus. In either case, he has not told a falsehood and therefore is telling the truth.

Conditional (cont)

Conditional—Truth Table

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Conditional (cont)

For a conditional $p \rightarrow q$, statement p is called the *hypothesis* and q is called the *conclusion*.*

A conditional is false *only* when the hypothesis is true and the conclusion is false.

Example: Constructing a Truth Table for a Conditional Statement

Construct a truth table for the statement
$(\sim p \vee q) \rightarrow (\sim p \wedge \sim q)$.

Solution

Because there are two variables, *p* and *q*, the truth table has four lines. As usual, we first consider the order of operations, as we indicate above the table.

(see next slide)

Example: Constructing a Truth Table for a Conditional Statement (cont)

			Step 1		Step 2	Step 3
p	q	$\sim p$	$\sim p \vee q$	$\sim q$	$\sim p \wedge \sim q$	$(\sim p \vee q) \rightarrow (\sim p \wedge \sim q)$
T	T	F	T	F	F	**F**
T	F	F	F	T	F	**T**
F	T	T	T	F	F	**F**
F	F	T	T	T	T	**T**

Use these columns to compute step 3.

Notice that in lines 1 and 3, there is a true hypothesis and a false conclusion,

Derived Forms of a Conditional

The converse, inverse, and contrapositive are three derived forms of a conditional.

We can derive the following statements from the conditional $p \to q$:

The converse has the form $q \to p$.

The inverse has the form $\sim p \to \sim q$.

The contrapositive has the form $\sim q \to \sim p$.

Derived Forms of a Conditional

The following table will help you remember how to construct these derived forms of a conditional:

Name of Derived Form	**How it is Constructed**
Conditional	$p \rightarrow q$
Converse	Switch p and q.
Inverse	Negate both p and q. Don't switch.
Contrapositive	Negate both p and q and also switch.

Example: Rewriting the Converse, Inverse, and Contrapositive of Statements in Words

Write, in words, the converse, inverse, and contrapositive of the statement:

"If marijuana is legalized, then drug abuse will increase."

Solution

To understand this problem better, we first write this statement symbolically as $m \rightarrow d$, where m stands for "Marijuana is legalized." and d stands for "Drug abuse will increase."

Example: Rewriting the Converse, Inverse, and Contrapositive of Statements in Words (cont)

We next write each of the derived forms symbolically.

To form the converse, we ***switch*** *d* and *m* to get the form $d \rightarrow m$. Translating this into words gives us "If drug abuse will increase, then marijuana is legalized."

Example: Rewriting the Converse, Inverse, and Contrapositive of Statements in Words (cont)

To form the inverse, we ***negate*** both *m* and *d* to get the form $\sim m \rightarrow \sim d$, which we can write as

"If marijuana is not legalized, then drug abuse will not increase."

Example: Rewriting the Converse, Inverse, and Contrapositive of Statements in Words (cont)

To form the contrapositive, we ***negate*** both *m* and *d* and also ***switch*** to get the form
$\sim d \rightarrow \sim m.$

We write this as "If drug abuse will not increase, then marijuana is not legalized."

Example: Equivalence of the Derived Forms of a Conditional

Which of the statements $p \rightarrow q$, $q \rightarrow p$, $\sim p \rightarrow \sim q$, and $\sim q \rightarrow \sim p$ are logically equivalent?

Solution

We will answer this question by comparing the truth tables for these statements.

(see next slide)

Example: Equivalence of the Derived Forms of a Conditional

We can see from the table that the original conditional and its contrapositive are logically equivalent. Neither the converse nor the inverse is equivalent to the original conditional; however, they are equivalent to each other.

logically equivalent

p	q	$p \rightarrow q$ Conditional	$q \rightarrow p$ Converse	$\sim p$	$\sim q$	$\sim p \rightarrow \sim q$ Inverse	$\sim q \rightarrow \sim p$ Contrapositive
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

logically equivalent

Alternative Wording of Conditionals

q if *p*	Here the *if* still is associated with *p* even though it occurs later in the sentence.
p only if *q*	Recognize that *only if* does not say the same thing as *if.* The *if* condition is the hypothesis; the *only if* condition is the conclusion.
p is sufficient for *q.*	The sufficient condition is the hypothesis.
q is necessary for *p.*	The necessary condition is the conclusion.

Example: Rewriting Statements in “If … Then” Form

Write each statement in *if…then* form.

a) Your driver’s license will be suspended if you are convicted of driving under the influence of alcohol.

b) You will graduate only if you have a 2.5 grade point average.

c) To hold your reservation, it is sufficient to give us your credit card number.

d) To qualify for a discount on your airline tickets, it is necessary to pay for them two weeks in advance.

Example: Rewriting Statements in "If … Then" Form (cont)

Solution

a) Your driver's license will be suspended if you are convicted of driving under the influence of alcohol.

Because the *if* goes with the clause "you are convicted of driving . . . ," that clause is the hypothesis.

We can rewrite this sentence as "If you are convicted of driving under the influence of alcohol, then your driver's license will be suspended."

Example: Rewriting Statements in "If … Then" Form (cont)

b) You will graduate only if you have a 2.5 grade point average.

The *only if* goes with the clause "you have a 2.5 grade point average"; therefore, this is the conclusion.

We write this sentence as "If you graduate, then you have a 2.5 grade point average."

Example: Rewriting Statements in “If … Then” Form (cont)

c) To hold your reservation, it is sufficient to give us your credit card number.

The phrase *it is sufficient* identifies the hypothesis.

We write the sentence as “If you give us your credit card number, then [we will] hold your reservation.”

Example: Rewriting Statements in “If … Then” Form (cont)

d) To qualify for a discount on your airline tickets, it is necessary to pay for them two weeks in advance.

The necessary condition is the conclusion. Rewriting this sentence, we get “If you qualify for a discount on your airline tickets, then you pay for them two weeks in advance.”

Biconditional

The biconditional means that two statements say the same thing.

We symbolize the biconditional as $p \leftrightarrow q$.

The biconditional is true when both *p* and *q* have the same truth value.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Example: Computing a Truth Table for a Complex Biconditional

Construct a truth table for the statement
$\sim(p \vee q) \leftrightarrow (\sim q \wedge p)$.

			Step 1		Step 2	Step 3
p	q	$p \vee q$	$\sim(p \vee q)$	$\sim q$	$\sim q \wedge p$	$\sim(p \vee q) \leftrightarrow (\sim q \wedge p)$
T	T	T	F	F	F	T
T	F	T	F	T	T	F
F	T	T	F	F	F	T
F	F	F	T	T	F	F