Coloring $k$-trees with forbidden monochrome or rainbow triangles

Julian Allagan & Vitaly Voloshin

Department of Mathematics,
University of North Georgia, Watkinsville, Georgia
email: julian.allagan@ung.edu

&

Department of Mathematics and Geomatics,
Troy University, Troy, Alabama
email: vvoloshin@troy.edu

Abstract

An $(\mathcal{H}, H)$-good coloring is the coloring of the edges of a (hyper)graph $\mathcal{H}$ such that no subgraph $H \subseteq \mathcal{H}$ is monochrome or rainbow. Similarly, we define an $(\mathcal{H}, H)$-proper coloring being the coloring of the vertices of $\mathcal{H}$ with forbidden monochromatic and rainbow copies of $H$. An $(\mathcal{H}, K_i)$-good coloring is also known as a mixed Ramsey coloring when $\mathcal{H} = K_n$ is a complete graph, and an $(\mathcal{H}, K_i)$-proper coloring is a mixed hypergraph coloring of a $t$-uniform hypergraph $\mathcal{H}$. We highlight these two related theories by finding the number of $(T_{nk}, K_3)$-good and proper colorings for some $k$-trees, $T_{nk}$ with $k \geq 2$. Further, a partition of an edge/vertex set into $i$ nonempty classes is called feasible if it is induced by a good/proper coloring using $i$ colors. If $r_i$ is the number of feasible partitions for $1 \leq i \leq n$, then the vector $(r_1, \ldots, r_n)$ is called the chromatic spectrum. We investigate and compare the exact values in the chromatic spectrum for some 2-trees, given $(T_{2n}^n, K_3)$-good versus $(T_{2n}^n, K_3)$-proper colorings. In particular, we found that when $G$ is a fan, $r_2$ follows a Fibonacci recurrence.

Keywords: chromatic spectrum, Stirling numbers, mixed hypergraphs, $k$-trees.

1 Preliminaries

It is customary to define a hypergraph $\mathcal{H}$ to be the ordered pair $(X, \mathcal{E})$, where $X$ is a finite set of vertices with order $|X| = n$ and $\mathcal{E}$ is a collection of nonempty subsets of $X$, called (hyper)edges. $\mathcal{H}$ is said to be linear (otherwise it is nonlinear) if $E_1 \cap E_2$ is either empty or a singleton, for any pair of hyperedges. The number of vertices contained in $E$ of $\mathcal{E}$, denoted $|E|$, is the size of $E$. When $|E| = r$, $\mathcal{H}$ is said to be $r$-uniform and a 2-uniform hypergraph.
$H = G$ is a graph. For more basic definitions of graphs and hypergraphs, we recommend [17].

Consider the mapping $c : A \to \{1, 2, \ldots, \lambda\}$ being a $\lambda$-coloring of the elements of $A$. A subset $B \subseteq A$ is said to be monochrome if all of its elements share the same color and $B$ is rainbow if all of its elements have distinct colors. Let $H$ be a subgraph of a graph $G$. An edge coloring of $G$ is called $(G; H)$-good if it admits no monochromatic copy of $H$ and no rainbow copy of $H$. Likewise, a $(G; H)$-proper coloring is the coloring of the vertices of $G$ such that no copy of $H$ is monochrome or rainbow. Figure 1(A) is an example of a $(G; K_3)$-proper coloring while Figure 1(B) shows a $(G; K_3)$-good coloring.

Axenovich et al.[2] have referred to $(K_n, K_3)$-good coloring as mixed-Ramsey coloring, a hybrid of classical Ramsey and anti-Ramsey colorings [2, 8, 14] and the minumum and maximum numbers of colors used in a $(K_n, K_3)$-good coloring have been the subject of extensive research in [2, 3], for instance. Further, in mixed hypergraph colorings [16], a hypergraph $\mathcal{H}$ that admits an $(\mathcal{H}; H)$-proper coloring is called a bihypergraph when $H = K_t$, the complement of a complete graph on $t \geq 3$ vertices. We note here that, mixed hypergraphs are often used to encode partitioning constraints, and recently bihypergraphs have appeared in communication models for cyber security [11]. Although this paper focuses on graphs, it is worth noting that the results concern some linear and nonlinear bihypergraphs as well.

A partition of an edge/vertex set into $i$ nonempty classes is called feasible if it is induced by a good/proper coloring using $i$ colors. If $r_i$ is the number of feasible partitions for each $1 \leq i \leq n$, then the vector $(r_1, \ldots, r_n)$ is called the chromatic spectrum. The chromatic spectrum of mixed hypergraphs has been well studied by several researchers such as Král and Tuza [5, 6, 12, 13]. Here, we found the values in the chromatic spectrum for any $(G; H)$-good or $(G; H)$-proper colorings when $G$ is some non-isomorphic 2-trees, which are triangulated graphs, and $H$ is a triangle. A comparative analysis of these values is presented in our effort to establish some bounds. In the process, we found that when $G$ is a fan, $r_2$ follows a shifted Fibonacci recurrence. If we denote the falling factorial by $\lambda^i = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - i + 1)$, then the (chromatic) polynomial $P(G; H, \lambda) = P(G; H) = \sum_{i=1}^{n} r_i \lambda^i$, counts the number of colorings given some constraint on $H$, using at most $\lambda$ colors. This polynomial is commonly known in the case of vertex colorings of graphs with a forbidden monochrome subgraph $H \in \{K_2, K_t\}$ [4, 7, 15]. In this paper, we also presented this polynomial for $k$-trees with forbidden monochrome or rainbow $K_t$ for all $t \geq 3$. Here, the Stirling number of the second kind is denoted by $\left\{ n \atop k \right\}$; it counts the number of partitions of a set of $n$ elements into $k$ nonempty subsets. See Table 4 for some of its values. These notations and other combinatorial identities can be found in [10]. In Appendix, we present some arrays of the values of the parameters involved in this article; the zero entries are omitted in each table.

2 Chromatic polynomial of some $k$-trees

As a generalization of a tree, a $k$-tree is a graph which arises from a $k$-clique by 0 or more iterations of adding $n$ new vertices, each joined to a $k$-clique in the old graph; This process
generates several non-isomorphic \( k \)-trees. Figure 1 shows two non-isomorphic 2-trees on 6 vertices. \( K \)-trees, when \( k \geq 2 \), are shown to be useful in constructing reliable network in [9]. Here, we denote by \( T^n_k \), a \( k \)-tree on \( n + k \) vertices which is obtained from a \( k \)-clique \( S \), by repeatedly adding \( n \) new vertices and making them adjacent to all the vertices of \( S \). When \( k = 2 \), this particular 2-tree is also known as an \((n+1) \)-bridge \( \theta(1, 2, \ldots, 2) \). See Figure 1(B) when \( n = 4 \).

\[ \begin{align*} 
\text{(A) Fan graph, } F^4 \\
\text{(B) 5-bridge graph, } T^4_2 
\end{align*} \]

Figure 1: Two non-isomorphic 2-trees with a unique \((F^4; K_3)\)-proper 4-coloring and a unique \((T^4_2; K_3)\)-good 5-coloring

**Theorem 2.1.** The number of \((T^n_k; K_{k+1})\)-good colorings of a \( k \)-tree on \( n + 2 \) vertices is
\[
P(T^n_k; K_{k+1}) = \lambda (\lambda^k - 1)^n + \lambda \binom{k}{2}(\lambda^k - (\lambda - \binom{k}{2})^n + (\lambda^k - \lambda^k - \lambda)\lambda^n \\
\]

*Proof.* Given any coloring of \( T^n_k \), one of the following is true:

(i) \( S \) is monochromatic giving \( \lambda \) colorings. For each such coloring, there are \( \lambda^k - 1 \) ways to color the remaining \( k \) edges, that arise from each of the \( n \) vertices added, giving the first term.

(ii) \( S \) is rainbow giving \( \lambda |S| \) colorings. For each such coloring, there are \( \lambda^k - (\lambda - |S|)^k \) ways to color the remaining \( k \) edges of each of the \( n(k+1) \) cliques, giving the second term.

(iii) \( S \) is neither monochromatic nor rainbow giving \( \lambda |S| - \lambda |S| - \lambda \) colorings. For each such coloring, there are \( \lambda^k \) ways to color the remaining edges of each added vertex, giving the last term. The result follows from the fact that \( |S| = \binom{k}{2} \).

Using a similar argument as in the proof of Theorem 2.1 when \( |S| = k \), gives

**Theorem 2.2.** The number of \((T^n_k; K_{k+1})\)-proper colorings of a \( k \)-tree on \( n + 2 \) vertices is given by
\[
P(T^n_k; K_{k+1}) = \lambda (\lambda - 1)^n + \lambda^k k^n + (\lambda^k - \lambda^k - \lambda)\lambda^n \\
\]
Remark 1: When \( k = 2 \), observe from the proof of Theorem 2.1 that the number of \( (T_{n}^{2}; K_{3}) \)-good colorings are identical for non-isomorphic 2-trees. However, in the next section, we show that this is not the case for \( (T_{n}^{2}; K_{3}) \)-proper colorings.

### 3 Chromatic spectra of (monochrome and rainbow)-triangle free 2-trees

Here, we find and compare the values in the chromatic spectrum of some 2-trees. The next proposition is instrumental in expressing several formulas in the previous section into a falling factorial form, giving the chromatic spectral values.

**Proposition 3.1.** The equality

\[
\lambda(\lambda - 1)^{n} = \sum_{k=2}^{n+1} \left[ \sum_{s=0}^{n-k+1} (-1)^{s} \binom{n}{s} \left\{ \binom{n+1-s}{k} \right\} \right] \lambda^{k}
\]

holds for all \( n \geq 1 \).

**Proof.** Clearly,

\[
\lambda(\lambda - 1)^{n} = \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \lambda^{n+1-i}
\]

\[
= \sum_{i=0}^{n} (-1)^{i} \binom{n}{i} \left[ \sum_{k=1}^{n+1-i} \left\{ \binom{n+1-i}{k} \right\} \lambda^{k} \right]
\]

\[
= \sum_{k=1}^{n+1} \left[ \sum_{s=0}^{n-k+1} (-1)^{s} \binom{n}{s} \left\{ \binom{n+1-s}{k} \right\} \right] \lambda^{k}
\]

\[
= \sum_{s=0}^{n} (-1)^{s} \binom{n}{s} \left\{ \binom{n+1-s}{1} \right\} \lambda^{1}
\]

\[
+ \sum_{k=2}^{n+1} \left[ \sum_{s=0}^{n-k+1} (-1)^{s} \binom{n}{s} \left\{ \binom{n+1-s}{k} \right\} \right] \lambda^{k}
\]

(1)

The result follows from the fact that (1) is equal to zero. □

**Corollary 3.0.1.** The chromatic spectrum of any \( (T_{n}^{2}; K_{3}) \)-good coloring is \( (r_{2}, \ldots, r_{k}, \ldots, r_{n+1}) \), where

\[
r_{k} = 3^{n-k+1} \left[ \sum_{i=0}^{n-k+1} (-1)^{i} \binom{n}{i} \left\{ \binom{n+1-i}{k} \right\} \right], \quad k = 2, \ldots, n+1.
\]

**Proof.** The result follows from Theorem 2.1 when \( k = 2 \), and Proposition 3.1. □

Here is the analogous result in a \( (G; K_{3}) \)-proper coloring when \( G \) is the \( (n+1) \)-bridge graph \( \theta(1, 2, \ldots, 2) \) which was shown to be a 2-tree on \( n+2 \) vertices. For simplicity, let \( T_{2}^{\theta} = \theta(1, 2, \ldots, 2) \).
Corollary 3.0.2. The chromatic spectrum of any \((T_2^n; K_3)\)-proper coloring of a 2-tree on \(n + 2\) vertices is \((r'_2, \ldots, r'_k, \ldots, r'_{n+1})\), where

\[
r'_k = \begin{cases} 
\sum_{i=0}^{n-k+1} (-1)^i \binom{n}{i} \binom{n+1-i}{k} & \text{if } k \geq 3 \\
2^n + 1 & \text{otherwise.}
\end{cases}
\]

Proof. From Theorem 2.2 (when \(k = 2\)), we have \(P(T_2^n; K_3) = \lambda(\lambda - 1)^n + 2^n \lambda^2\), to which we apply Proposition 3.1. Also, observe that from (2) when \(k = 2\),

\[
\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} \binom{n+1-i}{2} = 1
\]

given the second statement.

Now, we take a closer look at another well-known 2-tree. Construct a graph \(G\) as follows: start with a triangle \(\{w_1, w_2, u_1\}\), and iteratively add \(n - 1\) new vertices, such that each additional vertex \(u_i\) is adjacent to the pair \(\{u_i, u_{i-1}\}\), for \(i = 3, \ldots, n + 1\), and \(u_2\) is adjacent to the pair \(\{u_1, w_2\}\). We denote \(G\) by \(F^n\), a fan on \(n + 2\) vertices and Figure 1(A) is an example when \(n = 4\). Further, from the construction, it is clear that \(F^n\) is also a 2-tree. Here, we color the vertices of \(F^n\), and recursively count the number of \((F^n; K_3)\)-proper colorings. To help illustrate this recursion, we present the next example.

Example 3.1. Chromatic spectrum of an \((F^4; K_3)\)-proper coloring

Consider the fan \(F^4\), obtained by iteratively adding \(n = 4\) vertices to a base edge \(\{w_1, w_2\}\) as shown in Figure 1(A). When \(n = 1\), it is clear that there are exactly \(2\lambda^2 + \lambda^2\) ways to color the vertices of the triangle \(\{w_1, w_2, u_1\}\) so that it is neither monochrome nor rainbow. The first and second terms count the cases when \((a)\) \(c(u_1) \neq c(u_2)\) and \((b)\) \(c(u_1) = c(u_2)\), respectively. When \(n = 2\), from \((a)\) it follows that for each such colorings, there are exactly two ways to color \(u_2\); either \(c(u_2) = c(u_1) \neq c(w_2)\) or \(c(u_2) = c(u_1) = c(w_2)\). Likewise from \((b)\), there are \(\lambda - 1\) ways to color \(u_2\) such that \(c(u_2) \neq c(u_1) = c(w_2)\). Together, we have

\[
P(F^2; K_3) = 2(2\lambda^2) + (\lambda - 1)\lambda^2 = \lambda^2[2 + (\lambda - 1)] + 2\lambda^2.
\]

As the terms in last expression of (3) are arranged so that the first term counts the case when \(c(u_1) \neq c(u_2)\) and the last term counts the case when \(c(u_1) = c(u_2)\), we can apply once again the same argument to the newly added vertex \(u_3\). Thus, we have

\[
P(F^3; K_3) = 2[\lambda^2(2 + (\lambda - 1))] + (\lambda - 1)[2\lambda^2] = \lambda^2[2 + 3(\lambda - 1)] + \lambda^2[2 + (\lambda - 1)].
\]

Similarly, by adding \(u_4\) to \(F^3\), we obtain from (4),

\[
P(F^4; K_3) = \lambda^2[2 + 5(\lambda - 1) + (\lambda - 1)^2] + \lambda^2[2 + 3(\lambda - 1)],
\]

5
Table 1 in Appendix shows some of the chromatic spectral values given a \((T_2^n; K_3)\)-good coloring, a \((T_2^n; K_3)\)-proper coloring and an \((F^n; K_3)\)-proper coloring when \(n = 1, \ldots, 6\). These values can be derived from Corollary 3.0.1, Corollary 3.0.2, and Corollary 3.1.1 respectively, for each coloring condition.
**Theorem 3.1.** The number of \((F^n; K_3)\)-proper colorings is

\[
P(F^n; K_3) = \sum_{0 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor} \phi(n, r) \lambda(\lambda - 1)^{r+1},
\]

where

\[
\phi(n, r) = \begin{cases} 
  a_n + a_n\left\lceil \frac{n+r}{2} \right\rceil & \text{if } r < \frac{n}{2} \\
  a_n \frac{n}{2} & \text{otherwise}
\end{cases}
\]

and the values of \(a_{i,j}\) satisfying, for \(0 \leq j \leq i \leq n\),

(i) \(a_{i,0} = 2\) and \(a_{1,1} = 1\)

(ii) for all even \(i \geq 2\), \(a_{i,j} = \begin{cases} 
  a_{i-1,j} + a_{i-1,j+\left\lceil \frac{i+1}{2} \right\rceil} & ; 1 \leq j \leq \left\lceil \frac{i-1}{2} \right\rceil \\
  a_{i-1,j-\left\lceil \frac{i+1}{2} \right\rceil} & ; \left\lceil \frac{i+1}{2} \right\rceil \leq j \leq i
\end{cases}\)

(iii) for all odd \(i \geq 3\), \(a_{i,j} = \begin{cases} 
  a_{i-1,j} + a_{i-1,j+\left\lceil \frac{i+1}{2} \right\rceil} & ; 1 \leq j \leq \left\lceil \frac{i-1}{2} \right\rceil \\
  a_{i-1,j-\left\lfloor \frac{i}{2} \right\rfloor} & ; \left\lfloor \frac{i}{2} \right\rfloor \leq j \leq i
\end{cases}\)

**Proof.** When \(n = 1\), it follows that \(P(F^1; K_3) = \phi(1,0)\lambda(\lambda - 1)^1 = [a_{1,0} + a_{1,1}]\lambda(\lambda - 1)^1 = 3\lambda(\lambda - 1)\), since \(a_{1,0} = 2\) and \(a_{1,1} = 1\) by condition (i). For \(n \geq 2\), at each iteration, we separate the cases when \(c(u_1) \neq c(u_k)\) from when \(c(u_1) = c(u_k)\). Further, we rearrange the terms of the resulting expression of \(P(F^k; K_3)\) so that the first counts the colorings \(c(u_1) \neq c(u_k)\), and the last counts the colorings \(c(u_1) = c(u_k)\) for \(k = 1, \ldots, n\). Hence, for \(n \geq 1\),

\[
P(F^n; K_3) = \lambda^2 \left( \sum_{1 \leq k \leq \left\lceil \frac{n+1}{2} \right\rceil} a_{n,k-1}(\lambda - 1)^{k-1} \right) + \lambda^2 \left( \sum_{1+\left\lceil \frac{n+1}{2} \right\rceil \leq k \leq n} a_{n,k-1}(\lambda - 1)^{k-\left\lceil \frac{n+1}{2} \right\rceil-1} \right)
\]

\[
= \sum_{1 \leq k \leq \left\lceil \frac{n+1}{2} \right\rceil} [a_{n,k-1} + a_{n,\left\lceil \frac{n+1}{2} \right\rceil+k-1}]\lambda(\lambda - 1)^{k+1},
\]

where the coefficients \(a_{i,j}\) are obtained recursively from items (i) – (iii). By letting \(a_{i,j} = 0\) when \(i < j\), it follows that

\[
P(F^n; K_3) = \sum_{0 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor} \phi(n, r) \lambda(\lambda - 1)^{r+1},
\]

where

\[
\phi(n, r) = \begin{cases} 
  a_n + a_n\left\lceil \frac{n+r}{2} \right\rceil & \text{if } r < \frac{n}{2} \\
  a_n \frac{n}{2} & \text{otherwise}
\end{cases}
\]

\[
\boxed{\text{Proof end.}}
\]
Observation 1: The previous result can be reinterpreted as follows: Let $a_{0,0} = 2$ and define an $(n + 1) \times (n + 1)$ matrix $A$ whose entries are the coefficients $a_{i,j}$ for $0 \leq i, j \leq n$. It follows that (10) is equivalent to the equation $P = \lambda A \cdot B$, where

$$P = \begin{bmatrix} P(F^0; K_3) + \lambda(\lambda - 2) \\ P(F^1; K_3) \\ \vdots \\ P(F^n; K_3) \end{bmatrix}, \quad A = \begin{bmatrix} a_{0,0} \\ a_{1,0} \\ \vdots \\ a_{n,0} \end{bmatrix} \begin{bmatrix} a_{1,1} \\ \vdots \\ a_{n,1} \end{bmatrix} \ldots \begin{bmatrix} \ldots \\ a_{n,n} \end{bmatrix}$$

$$B = [B_1|B_2]^T \text{ with } B_1 = [(\lambda - 1)^1 \ldots (\lambda - 1)^{\lceil \frac{n+1}{2} \rceil}] \text{ and } B_2 = [(\lambda - 1)^1 \ldots (\lambda - 1)^{\lfloor \frac{n+1}{2} \rfloor}]$$

When $n = 10$, we present the entries of the lower triangular matrix $A$ in Table 2 to help in the verification of the formula. The matrix $A$ has several interesting properties some of which we discuss in the next observation. For now, it is easy to see that its determinant

$$\det(A) = \prod_{i=0}^{n} a_{i,i} = 2(\lceil \frac{n+1}{2} \rceil)!$$

and its characteristic polynomial is given by

$$(-1)^{n+1}(x - 1)^{\lceil \frac{n}{2} \rceil}(x - 2)^2(x - 3) \ldots (x - \lceil \frac{n+1}{2} \rceil)$$

Corollary 3.1.1. The values in the chromatic spectrum of any $(F^n; K_3)$-proper coloring are given by $r''_k = \sum_{k-2 \leq r \leq \lceil \frac{n}{2} \rceil} \phi(n, r) \sum_{0 \leq i \leq r-k+2} (-1)^i \binom{r+1}{i} \binom{r+2-i}{k}$, for each $k = 2, \ldots, \lceil \frac{n+1}{2} \rceil + 1$, with

$$\phi(n, r) = \begin{cases} a_{n,r} + a_{n,\lceil \frac{n+1}{2} \rceil} \quad & \text{if } r < \lceil \frac{n}{2} \rceil \\ a_{n,\lceil \frac{n}{2} \rceil} & \text{otherwise} \end{cases}$$

Proof. For each $r = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, we apply Proposition 3.1 to $P(F^n; K_3)$, giving

$$P(F^n; K_3) = \phi(n, 0)[(-1)^0 \binom{1}{0} \binom{2}{2}]\lambda^2 + \phi(n, 1)[(-1)^0 \binom{2}{0} \binom{3}{2} + (-1)^1 \binom{2}{1} \binom{2}{2}]\lambda^2 + \phi(n, 1)[-(-1)^0 \binom{2}{0} \binom{3}{3} \lambda^3 + \phi(n, 1)[(-1)^0 \binom{2}{0} \binom{3}{3}]\lambda^3 + \phi(n, 2)[(-1)^0 \binom{3}{0} \binom{4}{2} + (-1)^1 \binom{3}{1} \binom{3}{2} + (-1)^2 \binom{3}{2} \binom{2}{2}]\lambda^3 + \phi(n, 2)[-(-1)^0 \binom{3}{0} \binom{4}{4} \lambda^3 + (-1)^1 \binom{3}{1} \binom{4}{4} \lambda^3]$$
\[ + \phi(n, 3)[(-1)^0 \binom{3}{0} \binom{4}{4}] \lambda^4 \]

\[ + \phi(n, \lfloor \frac{n}{2} \rfloor)[(-1)^0 \binom{\lfloor \frac{n}{2} + 1 \rfloor}{0} \binom{\lfloor \frac{n + 1}{2} \rfloor + 1}{\lfloor \frac{n + 1}{2} \rfloor + 1}] \lambda^{\lfloor \frac{n + 1}{2} \rfloor + 1}. \tag{12} \]

Therefore,

\[ P(F^n; K_3) = \sum_{k=2}^{\lfloor \frac{n}{2} \rfloor + 1} \left( \sum_{r=k-2}^{\lfloor \frac{n}{2} \rfloor} \phi(n, r) \left[ \sum_{0 \leq i \leq r-k+2} (-1)^i \binom{r + 1}{i} \left\{ \binom{r + 2 - i}{k} \right\} \right] \lambda^k, \tag{13} \]

giving the result.

\[ \square \]

**Observation 2:** When \( k = \lfloor \frac{n+1}{2} \rfloor + 1 \), the last term of (13) is

\[ \phi(n, \lfloor \frac{n}{2} \rfloor) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 3 + \frac{n-1}{2} & \text{otherwise} \end{cases} \]

Also, it is worth noting that when \( k = 2 \),

\[ \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \phi(n, r) \left[ \sum_{0 \leq i \leq r} (-1)^i \binom{r + 1}{i} \binom{r + 2 - i}{2} \right] = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \phi(n, r); \text{this proceeds from the simple fact that} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+1-i}{2} = 1, \text{for all } n. \]

Further, observe that if we define \( b_i = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \phi(n, j) \) for each \( i \leq n \), it follows that \( b_i = \sum_j a_{i,j} \) and the sequence \( \{b_n\} \) satisfies the shifted Fibonacci recurrence given by \( b_0 = 2, b_1 = 3 \) and \( b_n = b_{n-1} + b_{n-2} \), for \( n \geq 2 \). From this observation, we determine the generating function in the next proposition.

**Proposition 3.2.** The number of partitions of the \( n+2 \) vertices of a fan into 2 nonempty classes such that no triangle is monochrome or rainbow is given by

\[ b_n = \frac{1}{\sqrt{5}}[(2 + \sqrt{5})\alpha^n - (2 - \sqrt{5})\beta^n], \text{where } \alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2} \]
Proof. Let \( b(x) = \sum_{n=0}^{\infty} b_n x^n \) such that \( b_0 = 2 \), \( b_1 = 3 \) and \( b_n = b_{n-1} + b_{n-2} \). It follows that

\[
\begin{align*}
\text{b}(x) &= 2 + 3x + \sum_{n=2}^{\infty} b_n x^n \\
&= 2 + 3x + x \sum_{k=1}^{\infty} b_k x^k + x^2 \sum_{k=0}^{\infty} b_k x^k \\
&= 2 + 3x + x (\sum_{k=0}^{\infty} b_k x^k - 2) + x^2 \sum_{k=0}^{\infty} b_k x^k \\
&= 2 + x + xb(x) + x^2 b(x).
\end{align*}
\]

This implies that \( b(x) = \frac{2 + x}{1 - x - x^2} = \frac{2 + x}{(x + \alpha)(x + \beta)} \), with \( \alpha = \frac{1 + \sqrt{5}}{2} \) and \( \beta = \frac{1 - \sqrt{5}}{2} \).

Using a partial fraction decomposition, and subsequently the power series, we obtain

\[
\begin{align*}
\text{b}(x) &= \frac{1}{\sqrt{5}} \left[ \frac{\beta - 2}{x + \beta} - \frac{\alpha - 2}{x + \alpha} \right] \\
&= \frac{1}{\sqrt{5}} \left[ \frac{\beta - 2}{\beta} (\sum_{n=0}^{\infty} \alpha^n x^n) - \frac{\alpha - 2}{\alpha} (\sum_{n=0}^{\infty} \beta^n x^n) \right] \\
&= \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left[ \frac{\beta - 2}{\beta} \alpha^n - \frac{\alpha - 2}{\alpha} \beta^n \right] x^n,
\end{align*}
\]

giving that \( b_n = \frac{1}{\sqrt{5}} \left[ \frac{\beta - 2}{\beta} \alpha^n - \frac{\alpha - 2}{\alpha} \beta^n \right] \). The result follows, after a simplification.

\( \square \)

In summary, the extreme chromatic spectral values given the aforementioned colorings are clear; the lower values are, \( r_2 = 3^n \), \( r'_2 = 2^n + 1 \), \( r''_2 = b(x) \) where

\[
b(x) = \frac{1}{\sqrt{5}} \left[ \frac{\beta - 2}{\beta} \alpha^n - \frac{\alpha - 2}{\alpha} \beta^n \right].
\]

Also, for all \( n > 1 \), the upper values are also shown to be \( r_{n+1} = 3^n \), \( r''_{n+1} = 1 \), and \( r'''_{n+1} = \begin{cases} 1 & \text{if } n \text{ even} \\ 3 + \frac{n-1}{2} & \text{otherwise} \end{cases} \)

4 Conclusion and future work

To the best of our knowledge, the problem of finding the exact chromatic spectral values in a \((K_n, K_t)\)-good coloring remains open for all \( t \geq 3 \) and larger values of \( n \); this particular problem which was originally by one of the authors has greatly inspired this research. When \( G \) is a 2-tree, the findings in Corollaries 3.0.1, 3.0.2, and 3.1.1 suggest the existence of
some constant $c < 1$, such that $r_k^* = cr_k$ where $r_k$ and $r_k^*$ are the corresponding values in the chromatic spectra of a $(G; K_3)$-proper and a $(G; K_3)$-good coloring, respectively. For instance, $c = (\frac{1}{3})^n$ when $G$ is an $(n+1)$-bridge. Further work is needed to determine whether the values in the chromatic spectrum of a $(G; H)$-good coloring remain upper bounds for their counterparts in a $(G; H)$-proper coloring, given any other graph $G$ and some subgraph $H$.

Also, the original definition of a $(G; H)$-proper coloring can be extended to include more than one subgraph. For instance, a $(G; H_1, \ldots, H_m)$-proper coloring is the coloring of the vertices of $G$ such that no copy of (distinct) subgraphs $H_i$ is monochrome or rainbow, for $i = 1, \ldots, m$. As such, when $G = \mathcal{H}$ and $H_i = \overline{K}_{t_i}$, $\mathcal{H}$ is a non-uniform bihypergraph with hyperedges of size $t_i \geq 3$. Some related results concerning non-uniform bihypergraphs can be found in [1]. As a step in this direction for graphs, we propose the next lemma.

This lemma shows that the chromatic spectral values of any $(F^n; K_3, H)$-proper coloring are identical when $H \in \{K_{1,t}, C_4, P_3 \square P_2\}$, where $P_3 \square P_2$ is isomorphic to $\theta(1, 3, 3)$, and $K_{1,t}$ is a complete bipartite graph with parts sizes $1$ and $t \geq 2$.

**Lemma 4.1.** Any (monochrome and rainbow)-triangle free proper coloring of a fan on $n+2$ vertices is an $(F^n; K_3, H)$-proper coloring for each $H \in \{K_{1,t}, C_4, P_3 \square P_2\}$, with $\lfloor \frac{n+1}{2} \rfloor \leq t \leq n+1$.

*Proof.* Let $S = \{w_1, w_2, u_3, \ldots, u_n\}$ denote the set of rim vertices and let $S_1 = \{w_1, w_2, \ldots, u_{2r}\}$, for each $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$. Suppose $H = K_{1,t}$ and consider a coloring of $F^n$ such that $c(u_1) = c(v_1)$ for each $v_1 \in S_1$. If $F^n$ contains no monochrome and rainbow triangle, it must be that $c(u_1) \neq c(v_2)$ for each vertex $v_2 \in (S \setminus S_1)$. Pick any $v'_2 \notin S_1$, and by letting $S_1 \cup \{u_1, v'_2\}$ be the vertex set of the subgraph $K_{1,t} \subset F^n$, it is clear that $K_{1,t}$ is neither monochrome nor rainbow and the size of $S_1 \cup \{v'_2\}$ gives the lower bound of $t$. To obtain the upper bound of $t$, color each vertex $v \in S$ with the same color and let $c(u_1) \neq c(v)$ for each $v \in S$. This gives an $(F^n; K_3)$-proper coloring and it is also an $(F^n; K_3, K_{1,t})$-proper coloring, where the vertex set of $K_{1,t} \subset F^n$ is $S \cup \{u_1\}$.

Now we show that any $(F^n; K_3)$-proper coloring is an $(F^n; K_3, C_4)$-proper coloring. Since every cycle on 4 vertices $C_4 \subset F^n$ must include $u_1$, assume that $C_4 = \{u_1, v_1, v_2, v_3, u_1\}$, an ordered sequence of vertices. If the set $\{u_1, v_1, v_2, v_3\}$ is monochrome/rainbow, then $C_4 \subset F^n$ contains a monochrome/rainbow triangle, which is impossible. Hence $C_4$ is neither monochrome nor rainbow, giving an $(F^n; K_3, C_4)$-proper coloring.

For all $n \geq 5$, observe that $H = P_3 \square P_2 \subset F^n$, and the argument follows from the fact that $C_4 \subset P_3 \square P_2$. \hfill \Box

In conclusion, it is worth noting that future work can address the coloring of the vertices of a graph with either forbidden monochrome subgraphs or forbidden rainbow subgraphs (but not both). As a step in this direction, we present a simple case when coloring the elements of an $n$-set such that no $t$-subset is rainbow.

**Corollary 4.0.2.** The chromatic spectral values in the colorings of the vertices of a complete graph $K_n$ such that no $K_t$ is rainbow are given by $r_k = \{n\}_k$, for $k = 1, \ldots, t - 1$.  

11
Note that these values also correspond to the chromatic spectral values of any complete $t$-uniform cohypergraph of order $n$; cohypergraphs are hypergraphs whose hyperedges are forbidden to be rainbow given any proper (vertex) coloring [16].

References


Appendix

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Table 1: chromatic spectral values of some \((G; K_3)\)-good colorings and some \((G; K_3)\)-proper colorings for \( n \leq 6 \)

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Table 2: Table of values of \( a_{i,j} \), which are the entries of the matrix \( A \) when \( n = 11 \)
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### Table 4: Table of values of $\binom{n}{k}$ when $n = 11$

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