

## Extending K-8 Mathematics Concepts in Alternate Bases

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### Abstract

When learning to represent mathematics with manipulatives, many pre-service K-8 teachers rely on memorized rote procedures to perform the associated mathematical tasks; then they arrange the manipulatives to match their result, often with minimal understanding of underlying mathematical connections. In a Number and Operations course for K-8 pre-service teachers, a portion of the class was conducted in alternate bases: Base 6 and Base 8 Blocks were used to model operations with integers to facilitate deeper understanding of the number systems and arithmetic processes being represented. Fractions and decimals were later covered only in Base 10. On midterm and final exams, students were tested not only on alternate base material covered in class, but on extensions of alternate base concepts that had not explicitly been covered. These extensions included performing arithmetic operations on integers too large to model with concrete manipulatives, as well identifying and computing with fraction and “decimal” representations of numbers in alternate bases. This paper describes the instructional tasks and assessment items used, as well as student outcomes on the assessments. Promising results suggest that with sufficiently deep understanding of a few core concepts, students can extend their mathematical thinking independently and meaningfully.

## Extending K-8 Mathematics Concepts in Alternate Bases

Research consistently suggests that elementary teachers' mathematics knowledge is significantly related to gains in student achievement (Hill, Rowan, & Ball, 2005). This relationship is supported by the finding that there is a strong association between teachers' mathematical knowledge and the mathematical quality of their instruction (Hill et. al, 2008). In particular, the quality of mathematics teaching and learning hinges on promoting understanding, rather than on leading students through cognitively undemanding activities such as recalling facts and applying well-rehearsed procedures (Boone, D'Ambrosio, & Harkness, 2004; Silver, Mesa, Morris, Star, & Benken, 2009; Slavin & Lake, 2008; Stigler & Heibert, 1999; cf. Bransford, Brown, & Cocking, 2000; Thompson, 2000; U.S. DoE, 2008).

Further, a discussion of teachers' mathematical content knowledge is not complete without qualifying the depth and nature of that knowledge. As one researcher suggests, although elementary teachers may "have a command of the facts and algorithms that comprise school mathematics, they lack a conceptual understanding of this mathematics" (Mewborn, 2001, p. 28). This finding has been echoed about elementary teachers in the U.S., in particular: When compared to those in China, elementary teachers in the U.S. had extremely limited conceptual understanding of the mathematics content they were responsible for teaching (Ma, 1999). Such conceptual understanding has been identified as key to improving mathematics instruction (Fennema & Franke, 1992; Lloyd & Wilson, 1998).

Consistent with these findings has been this author's observation that when learning to represent mathematics with manipulatives, many pre-service K-8 teachers rely on memorized rote procedures to perform the associated mathematical tasks; then they arrange the manipulatives to match their result, often with minimal understanding of underlying mathematical connections. The present study was conducted to explore the impact of requiring teacher candidates to use manipulatives to support their own learning of arithmetic operations in alternate bases, for which they could not rely on previously memorized algorithms. Among the outcomes examined were: 1) the teacher candidates' depth of understanding of number systems and arithmetic operations; and 2) their ability to extend this understanding to mathematical skills and contexts not explicitly covered in their instruction.

### Setting and Methodology

The study was conducted in a Number and Operations course for K-8 pre-service teachers. The catalog description of the course is

This course will emphasize the understanding & use of the major concepts of number and operations. Topics include problem-solving strategies; inductive and deductive reasoning; numeration systems and place value; operations and algorithms; identity elements and inverse operations; rational and irrational numbers; integers and number theory; special sets of numbers; exponents and decimals; ratios, percents and proportional reasoning.

The students referenced subsequently in this paper were teacher candidates— undergraduate students aspiring to be K-5 or middle grades teachers; thus, at no time in this paper does the term "student" refer to an elementary or middle grades student. There were 26 students registered for the course; 4 of these were majoring in middle grades education (MGED), and the other 22 were majoring in early childhood education (ECE). Two of the 26 students were male; the remaining 24 were female. Both of the male students were ECE majors.

The class met twice a week, totaling 3 in-class hours per week, over 13.5 weeks of instruction. Thus, there were 27 class meetings during the semester. For five of these class meetings, the focus of exploration was on the four whole number arithmetic operations— addition, subtraction, multiplication, and division. In each of these five class meetings, the majority of the class was conducted in alternate bases. Base 6 and Base 8 Blocks were used to model all four operations to facilitate deeper understanding of the number systems and arithmetic processes being represented. All questions posed with numbers in alternate bases were questions that could be modeled with manipulatives in the appropriate base; hence, all such questions used numbers of no more than 3 digits.

These five classes were held early in the semester, and the mid-term exam tested student understanding of these concepts, including their ability to perform these operations in alternate bases. However, the mid-term exam also included a bonus question that required students to extend the concepts of performing operations in an alternate base beyond the typical computations to which they had been exposed. The bonus question was as follows:

*Compute  $24153_6 + 13241_6$  and give the answer in Base 6.*

Because the numbers used in the bonus question were five digits, the values (and hence the operation) were not readily modeled with the manipulatives that students had previously used.

During the second half of the semester, fractions and decimals were covered, although these concepts were explored only in Base 10, using Pattern Blocks, Cuisenaire Rods, and Base 10 Blocks to help students represent the quantities and operations being investigated. Seven class meetings were devoted to operations on fractions, and two additional class meetings were used to explore operations on decimal numbers, all with an emphasis on representing each operation conceptually with manipulatives.

On the final exam, students were again given an opportunity to extend their conceptual knowledge beyond the concepts that had been explicitly covered in the class. These extensions were again posed as bonus questions, and they involved identifying and computing with fraction and “decimal” representations of numbers in alternate bases. The students had explored alternate bases only with whole numbers, and they had explored fractions only in Base 10. The series of bonus questions on the final exam was:

- a) *When we write fractions in other bases, both the numerator and denominator are given in the other base. Find the Base 10 equivalent of the Base 8 fraction  $\left(\frac{12}{31}\right)_8$*
- b) *Give the Base 10 equivalent of the Base 6 decimal fraction  $0.3_6$ , or  $\left(\frac{3}{10}\right)_6$*
- c) *Give the Base 10 equivalent of the Base 6 number  $152.3_6$*
- d) *Compute the product **in Base 6** of  $4.5_6 \times 2.1_6$*
- e) *Give the Base 10 equivalent of your Base 6 result from part (d).*

It should be noted that these items were given on a timed exam to students who had very likely never previously seen or considered how to interpret fractional representations in alternate bases. The items were deliberately scaffolded in such a way that students could gain sufficient familiarity with these novel ideas in the limited time available, and then possibly apply their understanding in that new context.

## Results

### *Midterm Exam*

All students attempted the bonus question on the midterm exam. Half of them (13 students) got the answer fully correct. Among the 13 students that got the answer incorrect, 6 of those demonstrated conceptual understanding of the algorithm but made arithmetic errors when adding the alternate base digits, resulting in an incorrect response.

Because students were always required to show work representing their thinking in this class, they also showed their thought process on the bonus question. It should be noted that the students who got the answer partly or fully correct routinely demonstrated conceptual understanding, specifically of the algorithm originally supported by Base 6 blocks. They did not convert the numbers to Base 10, perform the operations, and convert back to Base 6 at the end. Rather, the work they showed demonstrated the regrouping and exchanging necessary to perform the operation in Base 6 as given.

For example, some students drew a place value chart; they had grown accustomed to a convention of showing units, rods, and flats by drawing dots, lines, and squares, respectively. When they “ran out” of symbols to represent the increasing place values, they resorted to using other symbols in some cases, or to re-using symbols (such as dots) in other cases. In both cases, these students showed groupings of 6 figures in one column by circling a group of 6, then demonstrated moving the group to the next column by drawing an arrow. Still other students showed explicitly the values they “carried” from one place value column to the next (e.g., when adding the 4 and 5 in the “10’s” column, some students wrote 3, and then placed a 1 above the “100’s” column.) Among students who gave the correct answer, these expositions of their thinking gave strong evidence of conceptual understanding of the algorithm.

### *Final Exam*

Table 1 shows results of the series of bonus questions given on the final exam.

<i>Item</i>	<i>Not Attempted</i>		<i>Attempted but Incorrect</i>		<i>Attempted and Partially Correct</i>		<i>Attempted and Fully Correct</i>	
	n	%	n	%	n	%	n	%
(a)	8	30.8%	4	15.4%	1	3.8%	13	50.0%
(b)	8	30.8%	6	23.1%	1	3.8%	11	42.3%
(c)	7	26.9%	10	38.5%	5	19.2%	4	15.4%
(d)	6	23.1%	9	34.6%	7	26.9%	4	15.4%
(e)	9	34.6%	10	38.5%	5	19.2%	2	7.7%
<i>ALL 5</i>	2	7.7%	1	3.8%	22	84.6%	1	3.8%

Table 1. Ratings of Extension Items Given on Final Exam

As on the midterm exam, answers which reflected some conceptual understanding but which contained arithmetic or procedural errors were deemed partially correct; likewise, answers which displayed procedural knowledge applied incorrectly (i.e., with no apparent conceptual understanding) were rated as incorrect. The rating of “partially correct” was adopted only for purposes of this research; the only students who received “bonus points” for their answers were those whose answers were fully correct.

The final row in Table 1 shows the results for the series of questions as a whole. This information seems extremely relevant: It was not the same group of students who chose not to attempt every question. Rather, each question appears to have attracted a different group of students. Only 2 students chose not to attempt any items.

Table 1 also reveals that over 88% of students demonstrated conceptual understanding on at least one of the extension questions. This line of inquiry led to an investigation of how many items were rated partially or fully correct for each student. Table 2 addresses this question.

Number of Items	Number of Students Obtaining Fully Correct Rating		Number of Students Obtaining Partially Correct or Better	
	n	%	n	%
At least 1 item	14	53.8%	23	88.5%
At least 2 items	8	30.8%	17	65.4%
At least 3 items	3	11.5%	11	42.3%
At least 4 items	1	3.8%	6	23.1%
All 5 items	1	3.8%	2	7.7%

Table 2. Count of Students in Various Rating Categories

The analysis revealed that 14 students (53.8%) got at least one extension item fully correct. Because two of the extension items (a and b) were not considered especially demanding, more focus was given to the number of students who showed conceptual understanding on 3 or more items. Eleven students (42.3%) demonstrated some conceptual understanding on 3 or more items, though only 3 students obtained a rating of fully correct on 3 or more items. Also worth noting is that 6 students (23.1%) showed some conceptual understanding on 4 items or more.

### Discussion

It is clear that on both the midterm and final exams, at least half of the students demonstrated some ability to extend a mathematical concept they had learned beyond the context in which they learned it. One may argue that items (a) and (b) in the final exam sequence required little conceptual understanding, since they only required conversion of values from the given base to Base 10. However, even this conversion requires conceptual understanding of how numbers are constructed in other bases, and the placement of these numbers into a fraction representation is an extension to a new context, albeit an undemanding one.

One aspect of this study that deserves attention is how students' initial learning experiences may have facilitated their ability to extend their understanding to some new domain. The midterm extension was deemed to have approximately the same cognitive demand as item (c) in the sequence of extensions on the final exam. Yet far more students completed the midterm extension successfully than completed item (c) successfully on the final exam. Hence, the students demonstrated a greater ability to extend their understanding with the Base "n" addition algorithm than with representations of fractions. It may be reasonable to attribute this difference, at least in part, to the nature of the students' prior experiences learning both topics. Students learned the concepts needed for the midterm extension with the support of manipulatives *in an unfamiliar context*. That is, students learned to perform operations in an unfamiliar base primarily by using manipulatives, rather than learning how manipulatives could represent values and operations in a base with which they were already familiar. Students' conceptual understanding was supported by Base "n" Blocks, which were introduced in a setting that forced students to rely more fully on the manipulatives than they might have if the operations had

already been familiar to them (as in the case of fraction and decimal operations in the second half of the semester). It is also worth noting that the extension on the midterm paralleled the sort of conceptual extensions that these teachers will need to foster in their own classrooms: the extension of an arithmetic algorithm to numbers with digits increasingly high in place value.

By contrast, the extension items on the final exam combined the students' understanding of alternate bases (gained using manipulatives in an unfamiliar setting) with students' conceptual understanding of fractions and decimals (which, although presented and explored with manipulatives, were topics already familiar to the students). The transfer of mathematical "knowledge" to a new context seems to have been less complete under these conditions.

Because of the nature of this study, no control group was feasible. Therefore, the students taught using this method were not compared to students in other sections of the course. Hence, no specific conclusions may be drawn regarding the relative benefit of preparing pre-service teachers using methods such as these.

Another limitation of this study was that all items used to assess the extent of students' understanding were optional "bonus" items at the end of an exam. Students who did not feel the need for extra points may have been less motivated to attempt the items or to exercise much tenacity if they did attempt the items. Nevertheless, all students attempted the midterm extension, and all but 2 attempted some part of the extension sequence on the final exam.

In spite of the limitations noted, the results seem promising and support the notion that with sufficiently deep understanding of a few core concepts, students can extend their mathematical thinking independently and meaningfully.

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