Edge Cut Cycles and Cutting Numbers of Cycles and Graphs

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Fascinating Examples from Combinatorics, Discrete Mathematics, and Graph Theory

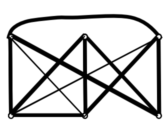
MathFest 2008

Agenda

- Introduction and Definitions
  -What is a "cutting number"?
- Foundation
  -Basic Results about cutting numbers
- Specific Structures
  -Cutting numbers of specific types of graph
- Min/Max Problems
  -Minimum or maximum edges for given cutting #

Introduction

- Imagine: Find a parade route through a city
  - Starts and ends at same place
  - Does not "disconnect" city when closed to traffic



Edges = Streets
Vertices = Intersections

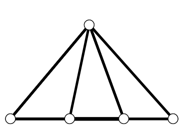
Definitions

- For a cycle C contained within a simple connected graph G, the ***cutting number of cycle C***, denoted C#(C,G), is the number of components in G – E(C).
- For a simple connected graph G, the ***cutting number of graph G*** is

C#(G) = max{C#(C,G) for all cycles C in G}

If G is acyclic, C#(G) = 0.

Example



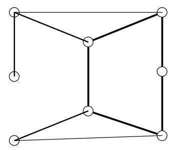
- Cycle with cutting number 1
- Cycle with cutting number 2
- Cycle with cutting number 3
- Therefore, **graph** has cutting number 3

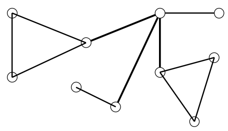
Notation and Assumptions

- All graphs considered are simple and connected unless otherwise stated.
- V(G) = vertex set of G
- *v*(G) = number of vertices in G
- E(G) = edge set of G
- e(G) = number of edges in G
- H ⊆ G ➔ H is a subgraph of G
- k(H) = number of components in H

Foundation (1)

If H ⊆ G, every component of G – E(H) contains a vertex of H.
(Recall G is connected.)

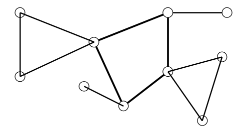


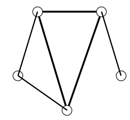


If H ⊆ G,
k(G – E(H)) ≤ *v*(H).

Foundation (2)

C#(C,G) ≤ *v*(C).



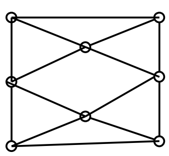


3 > 6 – 5 + 1

If G has a cycle C on more than *e*(G) – *v*(G) + 1 vertices, then C#(C,G) ≥ 2.
(*Removing more than e(G)–v(G)+1 will leave less than v(G)–1 edges-- not enough for a connected graph.)*

Foundation (3)

If a graph G with *e*(G) ≤ 2·*v*(G) – 2 has a Hamiltonian cycle, then C#(G) ≥ 2.

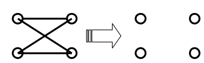


v(G) = 8, *e*(G) = 14

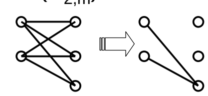
(Removing the cycle would leave less than v(G) – 1 edges-- not enough for a connected graph.)

Specific Structures: Complete Bipartite Graphs

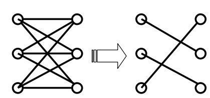
C#($K_{2,2}$) = 4



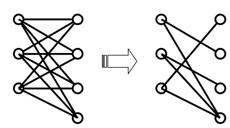
C#($K_{2,m}$) = 3 for m≥3



C#($K_{3,3}$) = 3

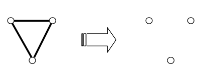


C#($K_{n,m}$) = 1 for n≥3, m≥4

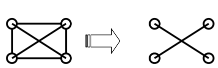


Specific Structures: Complete Graphs

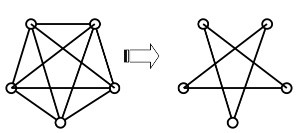
C#(K_3) = 3



C#(K_4) = 2

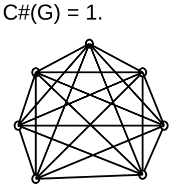


C#(K_n) = 1 for n ≥ 5

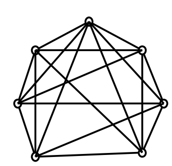


Specific Structures: Derivations from K_n

For n≥7, if G is formed by removing up to $\left\lfloor \frac{n}{2} \right\rfloor$ independent edges from K_n, then C#(G) = 1.



Graph G is
K_7 – 3 independent edges



No cycle will disconnect G if it is removed.

Specific Structures

If v(G) ≤ e(G) ≤ v(G) + 3, then C#(G) ≥ 2.

Proof Outline:

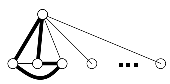
- If e(G) ≤ v(G) + 1, removing any cycle leaves too few edges for any connected graph
- Prove case for e(G) = v(G) + 2 by induction
  - K_4 is smallest feasible graph to consider
  - Proof relies on result for v(G) + 1 above
- Prove case for e(G) = v(G) + 3 by induction
  - Smallest feasible graph is v(G) = 5, e(G) = 8
  - Similar proof to above case and uses above result

Min/Max Problems Definitions

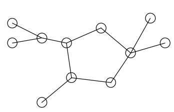
- m(k,n) is the minimum number of edges in a simple connected graph on n vertices with cutting number k
- M(k,n) is the maximum number of edges in a simple connected graph on n vertices with cutting number k

Results for m(k,n) – Minimum

m(2,n) = n+2 for n ≥ 4



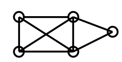
m(k,n) = n for 3 ≤ k ≤ n



M(2,n) – Maximum

For n ≥ 5, M(2,n) = $\binom{n-1}{2}+2$

- Proof by counting argument
- Define graph E_n as K_{n-1} + “ear”
- For n ≥ 7, E_n is only graph achieving M(2,n) edges
- Only one other graph for n=5,6



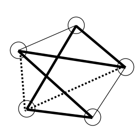
E_5

M(k,n) – Maximum

M(k-1,n) > M(k,n) for n ≥ 4 and 2 ≤ k ≤ n

Proof Outline:

- Induction on n – k
- Adding edges to maximized graph reduces cutting number



Questions