University of North Georgia
Department of Mathematics

Instructor: Berhanu Kidane
Course: College Algebra Math 1111
Text Book: For this course we use the free e–book by Stitz and Zeager with link:
Other online resources:
   e–book: http://msenux.redwoods.edu/IntAlgText/
   Tutorials: http://www.wtamu.edu/academic/anns/mps/math/mathlab/cr_algebra/index.htm

For more free supportive educational resources consult the syllabus
Chapter 6
Exponential and logarithmic Functions (Page 417)

Objectives: By the end of this chapter students should be able to:
- Identify Exponential and logarithmic Functions
- Identify graphs of exponential and logarithmic functions
- Sketches graphs of Exponential and Logarithmic functions
- Identify the relationship between exponential and logarithmic functions
- Identify and state rules of exponential and logarithmic functions
- Find domain and range of exponential and logarithmic functions
- Simplify exponential and logarithmic functions using their rules

Motivation

1) Interest: Compound

Compounded Continuously

Formulas:
\[ A = P \left(1 + \frac{r}{n}\right)^{nt} \] (Compound Interest)
\[ A = Pe^{rt} \] (Continuous Compounding)

\( A \) = Amount
\( P \) = Principal
\( r \) = Rate of interest (in %)
\( t \) = Time (usually in years)
\( n \) = Number of times amount is compounded

2) Radioactive Decay & Population Growth

Radioactive Decay: If \( m_0 \) is the initial mass of a radioactive substance with half life \( h \), then the mass \( m(t) \) remaining at time \( t \) is modeled by the function
\[ m(t) = m_0 e^{-rt}, \text{ where } r = \frac{ln2}{h} \]

Population Growth: A population that experiences a population growth increases according to the model:
\[ n(t) = n_0 e^{rt} \]

where \( n(t) \) = Population at time \( t \), \( n_0 \) = Initial size of population, \( r \) = relative rate of growth (expressed as a proportion of the population), \( t \) = time.

Example: C-14 Dating. The burial cloth of an Egyptian mummy is examined to contain 59% of the C-14 it contained originally. How long ago was the mummy buried? (The half-life of C-14 is 5730 years)

Example: World Population. The population of the world was 5.7 billion in 1995, and the observed relative growth was 2% per year.
   a) By what year will the population have doubled?
   b) By what year will the population have tripled?
Compound Interest

Compound Interest is calculated by the formula:

\[ A(t) = P \left( 1 + \frac{r}{n} \right)^{nt} \]

**Example 4:** If $4000 is borrowed at a rate of 5.75% interest per year, compounded quarterly, find the amount due at the end of the given number of years. a) 4 years  b) 6 years  c) 8 years

For \( r = 1 \), the compound interest formula becomes \( A(t) = P \left( 1 + \frac{1}{n} \right)^{nt} \).

**The Number \( e \)**

Consider the expression \( \left( 1 + \frac{1}{n} \right)^n \). We would like to investigate the value that this expression gets close to if \( n \) keeps getting larger. That is as \( n \to \infty \), \( \left( 1 + \frac{1}{n} \right)^n \to ? \)

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \left( 1 + \frac{1}{n} \right)^n )</th>
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<tbody>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>2.593742</td>
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<td>100</td>
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<tr>
<td>10^9</td>
<td>2.71828182709990</td>
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<tr>
<td>...</td>
<td></td>
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<tr>
<td>( \infty )</td>
<td>2.71828182845904...</td>
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</tbody>
</table>

From the **above table** we can make the following observation:

As \( n \) increases without bound \( \left( 1 + \frac{1}{n} \right)^n \) approaches the number \( e \), or equivalently

When \( n \to \infty \) the value \( \left( 1 + \frac{1}{n} \right)^n \to e \)
6.1 Exponential Functions

Exponential Functions of base \( a \)

**Definition:** An exponential function with base \( a \) is the function defined by \( f(x) = a^x \), where \( a > 0 \) and \( a \neq 1 \).

**Example 1:**

a) \( f(x) = 2^x \)

b) \( g(x) = \left(\frac{1}{2}\right)^x = 2^{-x} \)

c) \( f(x) = e^x \)

**Graphs of** \( f(x) = a^x \): there are two cases i) \( a > 1 \) and ii) \( 0 < a < 1 \)

Properties of the exponential function \( f(x) = a^x \):

1) The domain of \( f(x) = a^x \) is the set of all real numbers = \((−\infty, \infty)\)

2) The function \( f(x) = a^x \) is increasing for \( a > 1 \) and decreasing for \( 0 < a < 1 \)

3) The range of \( f(x) = a^x \) is \( \{y \mid y > 0\} = (0, \infty) \)

4) The function \( f(x) = a^x \) has y intercept \((0, 1)\) but has no x - intercept

5) The function \( f(x) = a^x \) is a one - to – one function, hence it is invertible.
Example 2: Sketch the graph of the following exponential functions:

a. \( f(x) = 2^x \)

b. \( f(x) = 0.8^x \)

c. \( f(x) = \sqrt[3]{3^x} \)

d. \( f(x) = \left(\frac{1}{2}\right)^x \)

e. \( f(x) = 10^x \)

f. \( f(x) = 0.6^x \)

Transformations:

Translations, Reflections, and Vertical and Horizontal Stretches and Shrinks

Translations:

1) Vertical Translation: \( y = f(x) \pm c, \text{ for } c > 0 \)
   - The graph of \( y = f(x) + c \) is the graph of \( y = f(x) \) shifted vertically \( c \) units up
   - The graph of \( y = f(x) - c \) is the graph of \( y = f(x) \) shifted vertically \( c \) units down

2) Horizontal Translations: \( y = f(x \pm c), \text{ for } c > 0 \)
   - The graph of \( y = f(x - c) \) is the graph of \( y = f(x) \) shifted horizontally \( c \) units to the right
   - The graph of \( y = f(x + c) \) is the graph of \( y = f(x) \) shifted horizontally \( c \) units to the left

Reflections

1) Across the x-axis:
   - The graph of \( y = -f(x) \) is the reflection of the graph of \( y = f(x) \) across the x-axis.

2) Across the y-axis:
   - The graph of \( y = f(-x) \) is the reflection of the graph of \( y = f(x) \) across the y-axis.

Stretches and Shrinks

Vertical Stretching and shrinking

To graph \( y = cf(x) \):
   - If \( c > 1 \), stretch the graph of \( y = f(x) \) vertically by a factor of \( c \)
   - If \( 0 < c < 1 \), shrink the graph of \( y = f(x) \) vertically by a factor of \( c \)

Horizontal Stretching and shrinking

To graph \( y = f(cx) \):
   - If \( c > 1 \), shrink the graph of \( y = f(x) \) horizontally by a factor of \( 1/c \)
   - If \( 0 < c < 1 \), stretch the graph of \( y = f(x) \) horizontally by a factor of \( 1/c \)

Example 3: Sketch the graph (Transformations of Exponential Functions)

a. \( f(x) = -2^x \)

b. \( f(x) = 2^x + 2 \)

c. \( f(x) = 2^{x - 1} \)

d. \( f(x) = -2^{x+1} - 2 \)
The Natural Exponential Function

**Definition:** The Natural Exponential Function is defined by \( f(x) = e^x \), with base \( e \).

**Continuously Compounded Interest**

**Example 1:** Continuously Compounded Interest is calculated by the formula:

\[
A(t) = Pe^{rt}
\]

Where \( A(t) = \) Amount after \( t \) years, \( P = \) Principal, \( r = \) Interest rate per year, and \( t = \) Number of years

**Example 2:** A sum of $5000 is invested at an interest rate of 9% per year compounded continuously

a) Find the value of \( A(t) \) of the investment after \( t \) years

b) Draw a graph of \( A(t) \)

**Laws of Exponents**

<table>
<thead>
<tr>
<th>Laws</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^1 = x )</td>
<td>( 6^1 = 6 )</td>
</tr>
<tr>
<td>( x^0 = 1 )</td>
<td>( 7^0 = 1 )</td>
</tr>
<tr>
<td>( x^{-1} = 1/x )</td>
<td>( 4^{-1} = 1/4 )</td>
</tr>
<tr>
<td>( x^m x^n = x^{m+n} )</td>
<td>( x^2 x^3 = x^{2+3} = x^5 )</td>
</tr>
<tr>
<td>( x^m / x^n = x^{m-n} )</td>
<td>( x^6 / x^2 = x^{6-2} = x^4 )</td>
</tr>
<tr>
<td>( (x^m)^n = x^{mn} )</td>
<td>( (x^2)^3 = x^{2\times3} = x^6 )</td>
</tr>
<tr>
<td>( (xy)^n = x^n y^n )</td>
<td>( (xy)^3 = x^3 y^3 )</td>
</tr>
<tr>
<td>( (x/y)^n = x^n / y^n )</td>
<td>( (x/y)^2 = x^2 / y^2 )</td>
</tr>
<tr>
<td>( x^{-n} = 1/x^n )</td>
<td>( x^{-3} = 1/x^3 )</td>
</tr>
</tbody>
</table>

And the Laws about Fractional Exponents:

<table>
<thead>
<tr>
<th>Laws</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^{1/n} = \sqrt[n]{x} )</td>
<td>( x^{1/3} = \sqrt[3]{x} )</td>
</tr>
<tr>
<td>( x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m )</td>
<td>( x^{2/3} = \sqrt[3]{x^2} = \left(\sqrt[3]{x}\right)^2 )</td>
</tr>
</tbody>
</table>

**Proof** of the law: \( x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m \) follows from the fact that \( m/n = m \times (1/n) = (1/n) \times m \)

**OER West Texas A&M University Tutorial 2:** [Integer Exponents](https://example.com) **Tutorial 5:** [Rational Exponents](https://example.com)
Consider the exponential function \( y = a^x, a > 0 \) and \( a \neq 1 \)

- \( y = a^x \) is a one-to-one function, thus it has an inverse
- The inverse of \( y = a^x \) is a function called the logarithmic function

Recall, the inverse of a function is obtained by interchanging the \( x \) and the \( y \) in the equation defining the function. Thus, the inverse of \( y = a^x \) is given by \( x = a^y \) which is the same as \( y = \log_a x \). That is we are saying \( x = a^y \iff y = \log_a x \)

**Graphically:** The graph of \( y = \log_a x \) obtained by reflecting the graph of \( y = a^x \) across the line \( y = x \).

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**Logarithmic Function, Base \( a \)**

**Definition:** (log function to any base \( a \))

\( y = \log_a x \) is the number \( y \) such that \( x = a^y \), where \( x > 0 \) and \( a > 0 \) and \( a \neq 1 \) is

**Examples**

a) **Case, \( a > 1 \):** \( y = \log_2 x, \ y = \log_3 x, y = \log_{1.3} x ; y = \log x; y = \ln x \)

b) **Case, \( 0 < a < 1 \):** \( y = \log_{1/2} x, \ y = \log_{1/3} x, y = \log_{0.4} x ; y = \log_{1/7} x \)
Graphs of $y = \log_a x$: Two cases i) $a > 1$ and ii) $0 < a < 1$

Properties of the logarithm function $f(x) = \log_a x$

1) The domain of $f(x) = \log_a x$ is $\{x \mid x > 0\} = (0, \infty)$

2) The function $f(x) = \log_a x$ is increasing for $a > 1$ and decreasing for $0 < a < 1$

3) The range of $f(x) = \log_a x$ is the set of all real numbers, in interval form $(-\infty, \infty)$

4) The function $f(x) = \log_a x$ has $x$ intercept $(1, 0)$ has no $y$ - intercept

5) The function $f(x) = \log_a x$ is a one - to – one function, hence it is invertible.

6) The function $f(x) = \log_a x$ is the inverse of the exponential function $y = a^x$ and vice versa

Example 1: Find graph the following logarithmic functions

a) $y = \log_3 x$, $y = \log_{1.3} x$ ; $y = \log x$; $y = \ln x$

b) $y = \log_{1/3} x$, $y = \log_{0.4} x$ ; $y = \log_{1/7} x$

Example 2: Find the domain and graph the following logarithmic functions

a) $y = -\log_3 x$

b) $y = \log_2(x - 2)$

c) $y = -\log_{0.2}(x + 1) + 2$

Example 2: Example 6.1.4. Page 425: Find the domain of the following functions

a) $f(x) = 2 \log(3 - x) - 1$

b) $g(x) = \ln \left(\frac{1}{x - 1}\right)$

Homework page 429: #1 – 74 (odd numbers)
Natural and Common Logarithms

**Definition:**
1) Logarithms with base $e$ are called **natural logarithms**.
   
   **Notation:** $\ln x$ used instead of $\log_e x$

2) Logarithms with base 10 are called **common logarithms**
   
   **Notation:** $\log x$ used instead of $\log_{10} x$

The calculator $\log$ is base 10, and the calculator $\ln$ is base $e$.

**Example 3:** Find using a calculator:

a) $\log 13$

b) $\log 10$

c) $\ln 9$

d) $\ln e$

e) $\log 5$

f) $\ln 5$

**Conversion between Exponential and Logarithmic Equations**

<table>
<thead>
<tr>
<th>Exponent Form</th>
<th>Logarithmic Form</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b^y = x$</td>
<td>$y = \log_b x$</td>
</tr>
<tr>
<td>$e^y = x$</td>
<td>$y = \ln x$</td>
</tr>
<tr>
<td>$10^y = x$</td>
<td>$y = \log x$</td>
</tr>
</tbody>
</table>

**Example 4:** Example 6.1.3 Page 424: Reading

**Examples 4:** Convert to the exponential form

a) $\log 1000 = 3$

b) $\log_3 81 = 4$

c) $\log 5 = b$

d) $\ln e = 1$

e) $\ln \sqrt[3]{e} = 1/3$

f) $\ln 9 = t$

**Example 5:** Convert each of the following to a logarithmic or exponential equation:

a) $16 = 2^x$

b) $\log_2 32 = 5$

c) $\log_3 9 = 2$

d) $7^2 = 49$

e) $10^{-3} = 0.001$

f) $x = \log_t M$

g) $\ln 4 = y$

h) $27^{1/3} = 3$
Properties of Logarithms (page 437)

OER West Texas A&M University Tutorial 44: Logarithmic Properties

1) \( \log_b(xy) = \log_b x + \log_b y \) (Product Rule)
2) \( \log_b(x/y) = \log_b x - \log_b y \) (Quotient Rule)
3) \( \log_b x^P = P \times \log_b x \) (Power Rule)
4) \( \log_b x = \frac{\log_c x}{\log_c b} \), for \( c > 0 \) and \( c \neq 1 \) (Change of Base)

If we change the base \( b \) to \( c = 10 \) or \( c = e \), then the change of base formula becomes:

\[ \log_b x = \frac{\log x}{\log b} \quad \text{OR} \quad \log_b x = \frac{\ln x}{\ln b} \]

5) Other properties: Let \( b > 0 \) and \( b \neq 1 \), then:
   a) \( \log_b 1 = 0 \), and so \( \ln 1 = 0 \)
   b) \( \log_b b = 1 \), and so \( \ln e = 1 \)
   c) \( \log_b b^x = x \), and so \( \ln e^x = x \)
   d) \( b^{\log_b x} = x \), and so \( e^{\ln x} = x \)

Example 1: Example 6.2.1 page 438: Reading

Example 1: Find each of the following using properties of log.
   a) \( \log 10000 \)          d) \( \log_7 49 \)
   b) \( \log_2 \left( \frac{1}{8} \right) \)       e) \( \log 100 \)
   c) \( \log_5 5^3 \)               f) \( \log_3 3 \)

Example 2: Find the values of the following using log properties
   a) \( \log_{10} 5 \)          c) \( \log^{\frac{3}{4}}42 \)
   b) \( \log_{10} 81 \)

Example 3: Simplify the following
   a) \( \left( 2^{\sqrt{5}} \right)^{\sqrt{20}} \)          c) \( \log_2(128/16) \)
   b) \( \log_2(\log_9 81) \)               d) \( e^{\ln^{\frac{3}{8}}11} \)

Example 4: Evaluate without a calculator whenever possible otherwise use calculator
   a) \( \log^{\frac{3}{4}}100 \)          c) \( \log_2 25 \)
   b) \( \log_3^{\frac{4}{3}}27 \)               d) \( \ln(\sqrt[3]{e^2}) \)

Example 5: Evaluate:
   a) \( \log_2 5 \)
   b) \( \log_{0.32} 99 \)
Example 6: Write as a single log:

a) \( \log_2(x - 2) + 3\log_2 x - \log_2(3 + x) \)

b) \( \log_b x + 2 \log_b y - 3 \log_b x \)

c) \( 2 \log_4 x + \log_4 y - \frac{1}{3}z \)

Example 7: Expand using log properties:

a) \( \log(3\sqrt{x}) \)

d) \( \log_b(x^2y^3z^2) \)

b) \( \log_5 \left( \frac{\sqrt{x + 1}}{9x^2(x-3)} \right) \)

e) \( \log_a \left( \frac{\sqrt[3]{a^2b}}{e^x} \right) \)

c) \( \log \left( \frac{\sqrt[x]{\frac{1}{y^2}}}{\sqrt[3]{z}} \right) \)

Homework page 445: #1 – 42 (odd numbers)

Solving Exponential and Equations Log Equations:

OER West Texas A&M University: Tutorial 45: \( \text{Exponential Equations}; \)

Tutorial 46: \( \text{Logarithmic Equations}; \)

Form | Strategy
--- | ---
1. \( b^x = b^y \) | Bases are the same, drop bases to obtain \( x = y \)
2. \( b^x = y \) | Take \( \log \) or \( \ln \) of both sides to change to the \( \log \) form
3. \( \log_b x = \log_b y \) | Bases are the same, drop the \( \log \)s to obtain \( x = y \)
4. \( \log_b x = y \) | Convert to exponential form to solve \( b^y = x \)

Example 1: Solve each of the following

a) \( 4^{3x} = 32x^{-2} \)

b) \( e^{x+3} = e^{x^2-4x} \)

c) \( 2^{5x} = 64 \)

d) \( 9x^2 \cdot 3^{5x} = 27 \)

e) \( 3^{x^2-5x} = \frac{1}{81} \)

Example 2: State the domain and solve the following

a) \( \log_2 x = 6 \)

b) \( \log_3 x + \log_3(2x - 3) = 3 \)

c) \( \log_3 x + \log_3(x + 1) = \log_3 2 \)

d) \( \log_2(x + 1) + \log_2(3x - 5) = \log_2(5x - 3) + 2 \)

e) \( \log_4 x + \log_4(x + 1) = \log_4 2 \)

f) \( \log(x + 2) - 3 \log 2 = 1 \)

g) \( \log_b 81 = -2 \)

Homework page 456: #1 – 33 (odd numbers)

Homework page 466: #1 – 24 (odd numbers)