Instructor: Berhanu Kidane

Course: College Algebra Math 1111

Text Book: For this course we use the free e – book by Stitz and Zeager with link:

Tutorials and Practice Exercises
- http://www.mathwarehouse.com/algebra/
- http://www.ixl.com/math/precalculus
- http://www.ltcconline.net/greenl/java/index.html

For more free supportive educational resources consult the syllabus
4.1 Rational Functions

Objectives: By the end of this section students should be able to:
- Define a rational function and give example
- Identify vertical, horizontal and oblique asymptotes
- Find vertical Horizontal and oblique asymptotes
- Sketch graphs of rational functions
- Solve application problems

Rational Function and Asymptotes

Definition: A function \( f(x) = \frac{p(x)}{q(x)} \), where \( q(x) \neq 0 \), \( p(x) \) and \( q(x) \) are polynomials is called a rational function.

Definition: (Domain)
The Domain of a rational function \( f \) is set of all inputs \( x \) for which \( q(x) \neq 0 \).
That is Domain of \( f = \{x : q(x) \neq 0\} \)

Example 4.1.1: Page 301
Example: Find the domain of:

a) \( f(x) = \frac{x+1}{x^2-4} \)
b) \( f(x) = \frac{x}{x^2-2} \)
c) \( f(x) = 1/x \)

A simple Rational Function

Example 1: Graph the function \( f(x) = \frac{1}{x} \)
The function \( f \) is not defined at \( x = 0 \), so, the domain of \( f = \{x : x \neq 0\} \)

The following two tables show that when \( x \) is close to zero, \( |f(x)| \) gets large

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>-10</td>
</tr>
<tr>
<td>-0.01</td>
<td>-100</td>
</tr>
<tr>
<td>-0.001</td>
<td>-1000</td>
</tr>
<tr>
<td>-0.0001</td>
<td>-10000</td>
</tr>
<tr>
<td>Approaches 0^-</td>
<td>Approaches to (-\infty)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>10</td>
</tr>
<tr>
<td>0.01</td>
<td>100</td>
</tr>
<tr>
<td>0.001</td>
<td>1000</td>
</tr>
<tr>
<td>0.0001</td>
<td>10000</td>
</tr>
<tr>
<td>Approaches 0^+</td>
<td>Approaches to (\infty)</td>
</tr>
</tbody>
</table>

We describe this behavior as follows
The first Table \( f(x) \rightarrow -\infty \) as \( x \rightarrow 0^- \); The second Table \( f(x) \rightarrow \infty \) as \( x \rightarrow 0^+ \)
The next two tables shows how $f(x)$ changes as $|x|$ becomes large

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Table 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$x$</td>
</tr>
<tr>
<td>-10</td>
<td>10</td>
</tr>
<tr>
<td>-100</td>
<td>100</td>
</tr>
<tr>
<td>-100000</td>
<td>100000</td>
</tr>
<tr>
<td>Approaches $-\infty$</td>
<td>Approaches $\infty$</td>
</tr>
</tbody>
</table>

The Tables shows that as $|x|$ becomes large, the value of $f(x)$ gets closer and closer to **zero**

That is:

$$f(x) \to 0 \text{ as } x \to -\infty \text{ and } f(x) \to 0 \text{ as } x \to -\infty$$

Using the information in these Tables and plotting few additional points, we obtain the graph of $y = 1/x$ as shown below

**Example 2:** Find the domain and sketch the graph using transformation properties on $f(x) = \frac{1}{x}$

a) $f(x) = \frac{2}{x-4}$ \hspace{1cm} **Answer D = $(-\infty, 4) \cup (4, \infty)$**

b) $y = \frac{x+1}{x-2}$

c) $y = \frac{1}{x+5}$
Example 3: In each of the following, which values of \( x \) may not be included in the domain? That is, which values are the singularities of the function? What is the domain of the function?

\( a) \quad y = \frac{1}{2x + 1} \)
\( b) \quad y = \frac{1}{x^2 - 16} \)
\( c) \quad f(x) = \frac{1}{x^2 + x - 6} \)

Asymptotes of Rational Functions

We consider three types of asymptotes: Vertical, Horizontal, and Oblique or Slant Asymptotes.

Arrow Notations:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \to a^- )</td>
<td>( x ) approaches ( a ) from the left</td>
</tr>
<tr>
<td>( x \to a^+ )</td>
<td>( x ) approaches ( a ) from the right</td>
</tr>
<tr>
<td>( x \to -\infty )</td>
<td>( x ) goes to negative infinity; ( x ) decreases without bound</td>
</tr>
<tr>
<td>( x \to \infty )</td>
<td>( x ) goes to infinity; ( x ) increases without bound</td>
</tr>
</tbody>
</table>

Definition of Vertical and Horizontal Asymptotes

1. **Vertical Asymptote (VA)** is a vertical line; that is a line perpendicular to the \( x \)– axis.
   - The line \( x = a \) is a Vertical Asymptote of the function \( y = f(x) \) if \( y \) approaches \( \pm \infty \) as \( x \) approaches \( a \) from the right or from the left.
   - Using Arrow Notations:
     - The line \( x = a \) is a Vertical Asymptote of the graph of a function \( y = f(x) \) if as \( x \to a^- \) or as \( x \to a^+ \), either \( f(x) \to \infty \) or \( f(x) \to -\infty \).

2. **Horizontal Asymptote (HA)** is a horizontal line; that is a line parallel to the \( y \)– axis.
   - The line \( y = b \) is a Horizontal Asymptote of the function \( y = f(x) \) if \( y \) approaches \( b \) as \( x \) approaches \( \pm \infty \).
   - Using Arrow Notations:
     - The line \( y = b \) is a Horizontal Asymptote of the graph of a function \( y = f(x) \) if as \( x \to -\infty \) or as \( x \to \infty \), \( f(x) \to b \).
Example 4: Vertical and Horizontal Asymptotes for the graph of \( y = \frac{1}{x} \), see fig below

**Question 1:** Where do we always find a vertical asymptote of a graph? At a singularity

**Question 2:** What does the equation of a vertical line look like? \( x = A \text{ number} \)

**Question 3:** What does the equation of a horizontal line look like? \( y = A \text{ number} \)

Example 5: Each of the following graphs is a translation of the graph of \( y = \frac{1}{x} \).

a) \( y = \frac{1}{x - 3} \)

What is the HA and VA?
b) \( y = \frac{1}{x+2} \)

What is the HA and VA?

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c) \( y = \frac{-1}{x+2} \) This is a reflection about the x-axis of graph b).

What is the HA and VA?

---

d) \( y = \frac{2x-5}{x-3} = \frac{1}{x-3} + 2 \)

What is the HA and VA?
Example 4.1.2: Page 306 vertical and horizontal asymptotes

Example 4.1.3: Page 307

Example 4.1.4: Page 309 List the horizontal asymptotes,

Example 5: Write the equation of the vertical and horizontal asymptote(s) of each of the following.

a) \( y = \frac{3x + 4}{x + 1} \)

b) \( y = \frac{1}{2x + 1} \)

c) \( y = \frac{1}{x^2 - 16} \)

d) \( f(x) = \frac{1}{x^2 + x - 6} \)

Oblique or Slant Asymptotes

3) Definition (Oblique or Slant Asymptote (OA))

When a linear asymptote is not parallel to the x- or y-axis, it is called an oblique asymptote or slant asymptote.

Using Arrow Notations

The line \( y = mx + b \) where \( m \neq 0 \) is called a slant or oblique asymptote of the graph of a function \( y = f(x) \) if as \( x \to \infty \) or as \( x \to -\infty \), \( f(x) \to mx + b \).

Example: Graph \( y = \frac{x^2 + 1}{x} \)

\[ y = x + \frac{1}{x} \]

Blue line is \( y = x \), the OA

Question 4: What does the equation of an oblique line look like?  
Ans. \( y = mx + b \)

Example 4.1.6: Page 312 Find the slant asymptotes

Example 6: Find the Oblique Asymptote: (Use long division to find the OA)

a) \( f(x) = \frac{2x^2 - 3x - 1}{x - 2} \)

b) \( f(x) = \frac{5x^3}{x^2 - 9} \)
Asymptotes Summary

Vertical Asymptote
First simplify the rational expression; then if \( a \) is a zero of the new denominator, then the line \( x = a \) is a vertical asymptote for the graph or the rational function.

**Example 1:** Find the vertical asymptotes:

a) \( f(x) = \frac{2}{x - 4} \)

b) \( f(x) = \frac{5x}{x^2 - 9} \)

c) \( f(x) = \frac{x^2 - 16}{x - 4} = \frac{(x-4)(x+4)}{x-4} = x + 4, \ x \neq 4 \)  
   No vertical asymptote

Horizontal Asymptote
Let \( f(x) = \frac{p(x)}{q(x)} = \frac{a_nx^n + \ldots + a_0}{b_kx^k + \ldots + b_0} \), \( n = \) degree of numerator, \( k = \) degree of denominator.

Then:

<table>
<thead>
<tr>
<th>Condition on degrees</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n &lt; k )</td>
<td>( y = 0 ) line is H.A.</td>
</tr>
<tr>
<td>( n = k )</td>
<td>( y = \frac{a_n}{b_k} ) line is the H.A</td>
</tr>
</tbody>
</table>

**Example 2:** Find the HA

a) \( f(x) = \frac{2}{x - 4} \)

b) \( f(x) = \frac{2x^3 + 3x - 7}{3x^3 - 5x^2 + 3x} \)

Oblique (or Slant) Asymptote
Let \( f(x) = \frac{p(x)}{q(x)} = \frac{a_nx^n + \ldots + a_0}{b_kx^k + \ldots + b_0} \), \( n = \) degree of numerator, \( k = \) degree of denominator.

Then:

<table>
<thead>
<tr>
<th>Condition on degrees</th>
<th>Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n &gt; k ) by exactly 1</td>
<td>( y = Q(x) ), the quotient poly is O.A.</td>
</tr>
<tr>
<td>( n &gt; k ) by more than 1</td>
<td>no H.A. or O.A.</td>
</tr>
</tbody>
</table>
Note: Graphs can cross horizontal or oblique asymptotes, but they cannot cross vertical asymptotes!

**Example 3:** Find the horizontal and/or oblique asymptotes:

a) \( f(x) = \frac{5x}{x^2 - 9} \)

- \( n = \text{degree of numerator} = 1 \)
- \( k = \text{degree of denominator} = 2 \)
- \( n < k \Rightarrow \text{Line } y = 0 \text{ is the horizontal asymptote} \)

b) \( f(x) = \frac{5x^4 + 3x^2 + 2x - 8}{2x^2 + 2x - 8} \)

- \( n = \text{degree of numerator} = 4 \)
- \( k = \text{degree of denominator} = 2 \)
- \( n > k + 1 \Rightarrow \text{No horizontal asymptote} \)

No oblique asymptote as well

**Example 4:** For each of the following functions fill the table with the correct asymptote(s) equation(s), otherwise write none if the function does not have the particular asymptote.

<table>
<thead>
<tr>
<th>Functions</th>
<th>Vertical Asymptote(s)</th>
<th>Horizontal Asymptote</th>
<th>Oblique Asymptote</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) = \frac{2}{x - 4} )</td>
<td>( x = 4 \text{ line} )</td>
<td>( y = 0 \text{ line} )</td>
<td>None</td>
</tr>
<tr>
<td>( f(x) = \frac{3x^2 + 1}{2x^2 - 4} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = \frac{x^4 + 2x^3 - 6x^2 - 2}{x^3 + 1} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = \frac{-x^2 + 3x}{2x^2 - 4x - 6} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( f(x) = \frac{3x^3 + 2x - 4}{x^2 - 4} )</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A) Find the **domain** and **range** and all **asymptotes** from the given graphs

1) \[ y = \frac{x^3 - 3x^2 + 2x + 2}{x^2 - 9} \]

2) \[ y = \frac{1}{x^2 + 1} \]
3) \[ y = \frac{3x^2 - x}{x^2 - 4} \]

4) \[ y = (-2)x(x-1)(2x-10)(3x-9)(2x+8)(3x+15). \] This is a polynomial function as well; find the degree, leading coefficient, constant term and zeros.
B) Sketch the graphs of the following rational functions.

a) \( f(x) = \frac{x}{x - 2} \)

b) \( f(x) = \frac{3x^2 + 1}{2x^2 - 4} \)

c) \( f(x) = \frac{3x^3 + 2x - 4}{x^2 - 4} \)

d) \( f(x) = \frac{-x^2 + 3x}{2x^2 - 4x - 6} \)

e) \( f(x) = \frac{5x}{x^2 - 9} \)

f) \( f(x) = \frac{x^2 - 16}{x - 4} \)

Homework Practice Problems
Exercises 4.1.1: Page 314 #1 – 21 (odd numbers),

**OER West Texas A&M** Tutorial 40: [Graphs of Rational Functions](https://www.youtube.com/watch?v=2N62v_63SBo)

Tutorial 41: [Practice Test on Tutorials 34 - 40](https://www.youtube.com/watch?v=P0ZggqB44Do4)

**Examples YouTube videos**
- Asymptotes of rational functions: [https://www.youtube.com/watch?v=2N62v_63SBo](https://www.youtube.com/watch?v=2N62v_63SBo)
- Finding vertical and horizontal asymptotes: [https://www.youtube.com/watch?v=P0ZggqB44Do4](https://www.youtube.com/watch?v=P0ZggqB44Do4)
- Rational functions graphs 1: [https://www.youtube.com/watch?v=ReEMqdZEEX0](https://www.youtube.com/watch?v=ReEMqdZEEX0)
- Graphs of rational functions 2: [https://www.youtube.com/watch?v=p7vcTWq6BFk](https://www.youtube.com/watch?v=p7vcTWq6BFk)