Chapter 9 – Functions of Several Variables

* 9.1 Concepts
  - you have already been exposed to functions of several independent variables

* 9.2 Graphical Representation - Skip

* 9.3 Partial Differentiation
  - you should already have an idea of what this entails but as a reminder:

$$z = f(x, y) = x^3 - 3xy^2$$

$$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial f}{\partial y} = -6xy$$

  - higher derivatives

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 - 3y^2\right) = 6x$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y} = \frac{\partial}{\partial x}\left(-6xy\right) = -6y$$

  - alternative notations

$$f_x = \frac{\partial f}{\partial x} \qquad f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} \qquad f_{xyz} = \frac{\partial^3 f}{\partial x \partial y \partial z}$$

$f(x_1, x_2, x_3)$: $\left(\frac{\partial f}{\partial x_1}\right)_{x_2, x_3}$ means that we are taking the derivative wrt x_1 while holding x_2 & x_3 constant

other derivatives: $\left(\frac{\partial f}{\partial x_2}\right)_{x_1, x_3}$ $\left(\frac{\partial f}{\partial x_3}\right)_{x_2, x_1}$

* 9.4 Stationary Points – Skip due to time

* 9.5 The Total Differential
  - cutting to the chase:

For a function, $z = f(x_1, x_2, \ldots, x_n)$

The total differential is: $dz = \left(\frac{\partial f}{\partial x_1}\right)dx_1 + \left(\frac{\partial f}{\partial x_2}\right)dx_2 + \ldots + \left(\frac{\partial f}{\partial x_n}\right)dx_n$

  - Example: $V = nRT / P$

$$dV = \left(\frac{\partial V}{\partial n}\right)dn + \left(\frac{\partial V}{\partial T}\right)dT + \left(\frac{\partial V}{\partial P}\right)dP$$

$$dV = \frac{RT}{P}dn + \frac{nR}{P}dT - \frac{nRT}{P^2}dP$$

* 9.6 Some Differential Properties

- Change of Independent Variables

let $z = f(x, y)$ with a total differential of

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

and let x & y also have dependent variables: $x(u,v)$ and $y(u,v)$

then z may also have this differential:

$$dz = \left(\frac{\partial z}{\partial u}\right)_v du + \left(\frac{\partial z}{\partial v}\right)_u dv$$

So how do we change between these two different sets?

- starting with the first equation, divide both side by ∂u while hold ν constant

$$\left(\frac{\partial z}{\partial u}\right)_v = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial u}\right)_v + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial u}\right)_v$$

- using the same type of logic we can get the remaining relationships

$$\left(\frac{\partial z}{\partial v}\right)_u = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial v}\right)_u + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial v}\right)_u$$

$$\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{\partial z}{\partial u}\right)_v \left(\frac{\partial u}{\partial x}\right)_y + \left(\frac{\partial z}{\partial v}\right)_u \left(\frac{\partial v}{\partial x}\right)_y$$

$$\left(\frac{\partial z}{\partial y}\right)_x = \left(\frac{\partial z}{\partial u}\right)_v \left(\frac{\partial u}{\partial y}\right)_x + \left(\frac{\partial z}{\partial v}\right)_u \left(\frac{\partial v}{\partial y}\right)_x$$

We can use these relationships to change between sets of coordinates.

Example: Given $z = f(x, y)$ with $x = a^2 b$ and $y = a \sin b$ find the partial derivatives for all variables in the system of equations.

$$\left(\frac{\partial z}{\partial a}\right)_b = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial a}\right)_b + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial a}\right)_b = \left(\frac{\partial z}{\partial x}\right)_y \cdot 2ab + \left(\frac{\partial z}{\partial y}\right)_x \cdot \sin b$$

$$\left(\frac{\partial z}{\partial b}\right)_a = \left(\frac{\partial z}{\partial x}\right)_y \left(\frac{\partial x}{\partial b}\right)_a + \left(\frac{\partial z}{\partial y}\right)_x \left(\frac{\partial y}{\partial b}\right)_a = \left(\frac{\partial z}{\partial x}\right)_y \cdot a^2 + \left(\frac{\partial z}{\partial y}\right)_x \cdot a \cos b$$

$x = a^2 b \rightarrow a = \sqrt{x / b}$ and $b = x / a^2$

$$\left(\frac{\partial z}{\partial x}\right)_y = \left(\frac{\partial z}{\partial a}\right)_b \left(\frac{\partial a}{\partial x}\right)_y + \left(\frac{\partial z}{\partial b}\right)_a \left(\frac{\partial b}{\partial x}\right)_y = \left(\frac{\partial z}{\partial a}\right)_b \cdot \frac{1}{2}\left(\frac{x}{b}\right)^{-1/2} \frac{1}{b} + \left(\frac{\partial z}{\partial b}\right)_a \cdot \frac{1}{a^2}$$

$y = a \sin b \rightarrow a = y / \sin b$ and $b = \sin^{-1} y / a$

$$\left(\frac{\partial z}{\partial y}\right)_x = \left(\frac{\partial z}{\partial a}\right)_b \left(\frac{\partial a}{\partial y}\right)_x + \left(\frac{\partial z}{\partial b}\right)_a \left(\frac{\partial b}{\partial y}\right)_x = \left(\frac{\partial z}{\partial a}\right)_b \cdot \frac{1}{\sin b} + \left(\frac{\partial z}{\partial b}\right)_a \frac{1}{\sqrt{1 - \left(\frac{y}{a}\right)^2}} \frac{1}{a}$$

* 9.7 Exact Differentials

- next semester after we learn the laws of thermodynamics we will see this eqn:

$$dU = TdS - PdV$$

- Since *dU* depends on *dS* and *dV* according to the previous eqn then *U(S,V)*

$$dU = \left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV$$

- A comparison of the two equations gives rise to 2 relationships:

$$\left(\frac{\partial U}{\partial S}\right)_V = T \text{ and } \left(\frac{\partial U}{\partial V}\right)_S = -P$$

- an exact differential is one who follows this example or:

for $F = F(x, y)$, $G = G(x, y)$ and $z = z(x, y)$

$$Fdx + Gdy = dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

exactness test: $\left(\frac{\partial G}{\partial x}\right)_y = \left(\frac{\partial F}{\partial y}\right)_x \rightarrow$ Euler reciprocity: $\frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x}$

- this idea will become very important when we do Maxwell relations next semester